Pseudo-Riemannian manifolds with many Killing spinors

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20 July 2009

Figueroa-O'Farrill, Meessen and Philip showed that M-theory backgrounds with more than 24 supersymmetries are locally homogeneous. Notice that

$$24 = (3/4) \cdot 32, \qquad N = 32 = \dim S_{1,10}.$$

This result is obtained from a careful analysis of the Killing spinor equations of M-theory. Inspired by this work, we study how many Killing spinors of a pseudo-Riemannian n-dimensional manifold (M, g) of signature s implies that M is locally homogeneous.

We define a $\wedge^k TM$-valued bilinear form

$$S \otimes S \to \wedge^k TM, \ (s, t) \mapsto [s, t]_k,$$

on the spinor bundle S which associates to a pair of conformal Killing spinors s, t a conformal Killing polyvector field $\omega = [s, t]_k$. For $k = 1$ we obtain conformal Killing vector fields.

The main results:

(M, g) pseudo-Riemannian n-manifold of signature s with a spinor bundle S of rank N.

- M is locally homogeneous if it admits more than $\frac{3}{4}N$ (independent) Killing spinors with the same Killing number,
  unless $n \equiv 1 \pmod 4$ and $s \equiv 3 \pmod 4$.

- M is locally homogeneous if it admits $k_\pm$ Killing spinors with Killing number $\pm\lambda$ such that $k_+ + k_- > \frac{3}{2}N$,
  unless $n \equiv s \equiv 3 \pmod 4$.

- Similarly, a pseudo-Riemannian manifold with more than $\frac{3}{4}N$ *conformal* Killing spinors is *conformally* locally homogeneous.

For definite metrics, the bounds $\frac{3}{4}N$ and $\frac{3}{2}N$ in the above results can be relaxed to $\frac{1}{2}N$ and N, respectively.

- a pseudo-Riemannnian spin manifold with more than $\frac{3}{4}N$ parallel spinors is flat and $\frac{1}{4}N$ parallel spinors suffice if the metric is definite.

- Similarly, a Riemannnian spin manifold with more than $\frac{3}{8}N$ Killing spinors with the Killing number $\lambda \in \mathbb{R}$ has constant curvature $4\lambda^2$.

- For Lorentzian or negative definite metrics the same is true with the bound $\frac{1}{2}N$.

- A Riemannian spin manifold with $\frac{3}{8}N$ Killing spinors with the Killing number $\lambda \in \mathbb{R} \setminus \{0\}$ can be locally represented in the form

  $$M = I \times M_1 \times M_2,\ g = ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2,$$

  where (M_1, g_1) is of constant curvature 1 or of dimension ≤ 1, (M_2, g_2) is a seven-dimensional 3-Sasakian manifold and $I \subset (0, \frac{\pi}{2})$ is an intervall,

- Finally, we give a description of (not necessarily complete) Riemannian manifolds admitting Killing spinors, which provides an inductive construction of such manifolds.

Conformally Killing polyvectors

Let (M, g) be an n-dimensional pseudo-Riemannian manifold.

Definition 1 *A k-vector field $\omega \in \Gamma(\wedge^k TM) \cong \Gamma(\wedge^k T^*M)$ ($k \geq 1$) is called Killing if*

$$X \lrcorner \nabla_X \omega = 0, \quad \text{for all} \quad X \in TM.$$

It is called conformally Killing if there exists a $(k-1)$-vector field $\tilde{\omega}$ such that

$$X \lrcorner \nabla_X \omega = g(X, X)\tilde{\omega}, \quad \text{for all} \quad X \in TM. \quad (1)$$

Characterisation of conformally Killing polyvectors

(i) $\omega \in \Gamma(\wedge^k TM)$ is Killing if and only if $\dot\gamma \lrcorner \omega$ is a parallel $(k-1)$-vector field along γ, for every geodesic γ:

$$\nabla_{\dot\gamma}(\dot\gamma \lrcorner \omega) = 0. \tag{2}$$

(ii) ω is conformally Killing if and only if $\nabla_{\dot\gamma}(\dot\gamma \lrcorner \omega) = 0$, for every null geodesic γ.

Killing spinors and conformal Killing spinors

Definition 2 *A spinor field $s \in \Gamma(S)$ is called Killing with Killing number $\lambda \in \mathbb{R}$ if*

$$\nabla_X s = \lambda X s, \quad \text{for all} \quad X \in TM,$$

where Xs is the Clifford product of the vector X and the spinor s.
It is called conformally Killing if there exists a spinor field $\tilde{s} \in \Gamma(S)$ such that

$$\nabla_X s = X\tilde{s}, \quad \text{for all} \quad X \in TM. \tag{3}$$

Remarks

1) The last equation implies

$$\tilde{s} = -\frac{1}{n} Ds, \tag{4}$$

where $Ds = \sum g^{ij} e_i \nabla_{e_j} s$ is the Dirac operator. In particular, any Killing spinor is an eigenspinor for the Dirac operator: $Ds = -n\lambda s$.

2) The Killing number is related to the scalar curvature by $scal = 4n(n-1)\lambda^2$. Therefore, the scalar curvature of a pseudo-Riemannian manifold with Killing spinor is constant and the Killing numbers of different Killing spinors on the same manifold coincide up to a sign. It is well known that a Riemannian manifold which admits a Killing spinor is Einstein, but this is no longer true for indefinite pseudo-Riemannian manifolds,

Endomorphism of S associated with a polyvector

Let $\gamma_v : S_p \to S_p$ be the Clifford multiplication with $v \in T_pM$.

The linear map $\gamma : \wedge^k T_pM \to \text{End}\,(S_p)$, for all $k \geq 1$,is defined by

$$\gamma_{v_1 \wedge \cdots \wedge v_k} := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) \gamma_{v_{\sigma 1}} \cdots \gamma_{v_{\sigma k}},$$

where $\mathfrak{S}_k$ is the symmetric group.

For $\lambda \in \wedge^0 T_pM = \mathbb{R}$ we put $\gamma_\lambda = \lambda \text{i}d \in \text{End}\ S_p$.

Admissible bilinear forms, their symmetry and type

Definition 3 *A bilinear form h on the spinor module satisfying*

$$\begin{aligned} h(s,t) &= \sigma h(t,s), \\ h(\gamma_X s,t) &= \tau h(t,\gamma_X s), \end{aligned} \tag{5}$$

for $s,t \in S$ and $X \in TM$, is called admissible of symmetry σ and type τ, where $\sigma,\tau \in \{-1,+1\}$.

There always exists a nondegenerate admissible bilinear form.

An admissible form is invariant under the connected spin group and defines a parallel section of $S^* \otimes S^*$. In the following, h always denote a parallel nondegenerate section of $S^* \otimes S^*$ of symmetry σ and type τ. Notice that

$$h(\gamma_\xi s,t) = \tau^k(-1)^{\frac{(k-1)k}{2}} h(s,\gamma_\xi t), \quad \xi \in \Gamma(\wedge^k TM).$$

Bilinear $\wedge^k TM$-valued form

We associate with $h(s,t)$ a bilinear $\wedge^k TM$-valued form $[s,t]_k$ on S defined by

$$g([s,t]_k, \xi) = h(\gamma_\xi s, t) \quad \forall \xi \in \Gamma(\wedge^k TM), s, t \in \Gamma(S). \tag{6}$$

(Here g is canonically extended to a nondegenerate symmetric bilinear form on the exterior algebra.) Such brackets occur in the classification of polyvector super-Poincaré algebras. For $k = 0$ we put $[s,t]_0 = h(s,t)$.

Conformal Killing polyvector associated with conformal Killing spinors

Theorem 1 *Let s, t be conformal Killing spinors on an n-dimensional pseudo-Riemannian spin manifold (M, g). Then $\omega = [s,t]_k \in \Gamma(\wedge^k TM)$ ($k \geq 1$) is a conformal Killing polyvector;*

$$X \lrcorner \nabla_X \omega = g(X,X)\tilde{\omega} \quad \forall X \in TM,$$

where $\tilde{\omega} \in \Gamma(\wedge^{k-1} TM)$ is given by

$$n\tilde{\omega} = (-1)^{k-1}[Ds,t]_{k-1} + \tau[s,Dt]_{k-1}. \qquad (7)$$

Corollary 1 *Let* s *and* t *be Killing spinors with Killing numbers* λ *and* μ*, respectively, and* $\omega = [s,t]_k$*. Then the following is true.*

- *(i)* ω *is a conformal Killing polyvector with* $\tilde{\omega} = (\lambda(-1)^k - \mu\tau)[s,t]_{k-1}$*.*
- *(ii) If* $\mu = (-1)^k \tau\lambda$*, then* $\omega = [s,t]_k$ *is a Killing polyvector.*
- *(iii) If* $\lambda = \mu = 0$ *then* ω *is parallel.*

Manifolds with many Killing spinors

Theorem 2 *Let* (M, g) *be a pseudo-Riemannian spin manifold of dimension* n*, signature* s *and spinor bundle* S *of rank* N*.*

(i) *If* (M, g) *admits* $k > \frac{3}{4}N$ *conformal Killing spinors, which are linearly independent at* $p \in M$*, then* (M, g) *admits* n *conformal Killing vector fields, which are linearly independent at* $p \in M$*.*

(ii) *Assume that* $n \not\equiv 1 \pmod 4$ *or* $s \not\equiv 3 \pmod 4$*. If* (M, g) *admits* $k > \frac{3}{4}N$ *Killing spinors with the same Killing number, which are linearly independent at* $p \in M$*, then* (M, g) *admits* n *Killing vector fields, which are linearly independent at* $p \in M$*.*

(iii) Assume that
$n \equiv 1 \pmod 4$ and $s \equiv 3 \pmod 8$.
Then S admits a parallel hypercomplex structure $J_1, J_2, J_3 = J_1 J_2 \in \Gamma(\mathrm{End}\, S)$, which commutes with Clifford multiplication. Let I be any complex structure on S which is a linear combination of J_1, J_2, J_3 with constant coefficients. If (M, g) admits $k > \frac{3}{4}N$ solutions $s \in \Gamma(S)$ of the equation

$$\nabla_X s = \lambda X I s, \quad \text{for all} \quad X \in TM, \qquad (8)$$

with the same $\lambda \in \mathbb{R}$, which are linearly independent at $p \in M$, then (M, g) admits n Killing vector fields, which are linearly independent at $p \in M$.

(iv) Assume that
$n \not\equiv 3 \pmod 4$ or $s \not\equiv 3 \pmod 4$.
If (M, g) admits k_+ Killing spinors with the Killing number λ, which are independent at p, and k_- Killing spinors with the Killing number $-\lambda$, which are independent at p, such that $k_+ + k_- > \frac{3}{2}N$, then it admits n Killing vector fields, which are independent at p.

(v) If g is definite, then (i)-(iv) hold under the weaker assumptions $k > \frac{1}{2}N$ and
$k_+ + k_- > N$, respectively.

Idea of the proof

There exists a parallel nondegenerate bilinear form h of symmetry σ and type $\tau = -1$ on S unless $n \equiv 1 \pmod 4$ and $s \equiv 3 \pmod 4$. (The Pin(n)-invariant scalar product on the spinor module associated with a positive definite scalar product, for instance, has $\tau = -1$.) For any conformal Killing spinors s, t, the vector field $[s,t]_1$ is conformal. Moreover, if s, t are Killing spinors with the same Killing number and $\tau = -1$, then $[s,t]_1$ is a Killing vector field. Therefore, to prove (i) and (ii) it suffices to show that

$$\Pi := [\cdot,\cdot]_1|_{S_0 \otimes S_0} : S_0 \otimes S_0 \to T_pM \qquad (9)$$

is surjective if the subspace $S_0 \subset S_p$ spanned by the values of the given (conformal) Killing spinors at p has dimension $> \frac{3}{4} \dim S_p$.

Suppose first that g is definite.
Then we have to show that
Π is surjective if $\dim S_0 > \frac{1}{2}\dim S_p$ or equivalently,
$\nexists v \in T_pM \setminus \{0\}$ such that

$$\gamma_v S_0 \subset S_0^\perp. \tag{10}$$

Suppose that there exists $v \in T_pM \setminus \{0\}$ s.t. (10) holds.
If $\dim S_0 > \frac{1}{2}\dim S_p$, then
$\dim S_0^\perp < \frac{1}{2}\dim S_p < \dim S_0$
and $\ker \gamma_v \neq 0$.
Since $\gamma_v^2 = -g(v,v)\mathrm{i}d$, this implies $g(v,v) = 0$ and $v = 0$. This proves the surjectivity of Π.
If g is indefinite, we can only conclude that v is a null vector.

Lemma 1 *For any non-zero null vector* v *the subspace* $L_v := \mathrm{ker}\,\gamma_v = \mathrm{im}\,\gamma_v \subset S_p$ *is* h*-isotropic of dimension* $\frac{1}{2}\mathrm{dim}\,S_p$*. Hence,*
$\mathrm{rk}\,\gamma_v = \frac{1}{2}\mathrm{dim}\,S_p$.

Consider the bilinear form $\beta = h(\gamma_v \cdot, \cdot)$ on S_p with $\operatorname{rk}\beta = \operatorname{rk}\gamma_v = \frac{1}{2}\dim S_p$.
Under the assumption $\gamma_v S_0 \subset S_0^{\perp}$, the matrix of β with respect to a basis adapted to a decomposition $S_p = S_0 \oplus S_1$ is given by

$$\begin{pmatrix} 0 & A \\ \sigma\tau A^t & B \end{pmatrix}$$

Therefore,

$$\frac{1}{2}\dim S_p = \operatorname{rk}\beta \leq \operatorname{rk} A + \operatorname{rk}(\sigma\tau A^t, B) \leq$$

$$2\dim S_1 = 2(\dim S_p - \dim S_0),$$

which implies

$$\dim S_0 \leq \frac{3}{4}\dim S_p.$$

We get a contradiction which shows that
$\Pi : S_0 \otimes S_0 \to T_pM$ is surjective.

Case when $\tau = 1$

Proposition 1 *Let h be a nondegenerate parallel bilinear form of symmetry σ and type $\tau = +1$ on the spinor bundle S of a pseudo-Riemannian spin manifold (M, g) and denote by $S(\lambda) \subset \Gamma(S)$ the vector space of Killing spinors with a given Killing number $\lambda \in \mathbb{R} \setminus \{0\}$. Then the image $[S(\lambda), S(\lambda)]_1 \subset \Gamma(TM)$ consists of Killing vector fields if and only if $S_0 := S(\lambda)|_p \subset S_p$ is totally isotropic for all $p \in M$ with respect to h. If S_0 is maximally isotropic at a point p then $[S(\lambda), S(\lambda)]_1 \neq 0$, hence, (M, g) admits a Killing vector field, which does not vanish at p.*

Remark

One can check that $[S_0, S_0]_1$ is one-dimensional for any maximally isotropic subspace S_0 of the spinor module $S_{2,3} = \mathbb{R}^4$ of Spin$(2,3)$. For the spinor module $S_{4,5}$ of Spin$(4,5)$ one can construct a maximally isotropic subspace S_0 such that $\dim[S_0, S_0]_1 = 4$.
These examples show that in general a vector space of Killing spinors spanning a maximally isotropic subspace of S_p for all p is not sufficient to produce a transitive Lie algebra of Killing fields.

A multiplicative invariant $\kappa(M,\lambda)$

Let $S(\lambda) = S(M,\lambda)$ be the vector space of Killing spinors with Killing number $\lambda \in \mathbb{R}$, $k := \dim S(\lambda)$ and

$$\kappa(M,\lambda) := \frac{k}{N}, \quad \kappa(M) := \kappa(M,0).$$

Then $\kappa(M) = 1$ if and only if M is flat and that $\kappa(M,\lambda) = \frac{\dim \mathbb{S}(\lambda)}{\mathrm{rk}\,\mathbb{S}}$,
where $\mathbb{S}$ is the complex spinor bundle and $\mathbb{S}(\lambda) = \mathbb{S}(M,\lambda)$ the vector space of complex Killing spinors with Killing number λ.

Proposition 2 *Let $M = M_1 \times M_2$ be the product of two pseudo-Riemannian spin manifolds. Then $\kappa(M) = \kappa(M_1)\kappa(M_2)$. In particular, $\kappa(M) = \kappa(M_1)$ if and only if M_2 is flat.*

Remark: The invariant $\kappa(M,\lambda)$ for $\lambda \neq 0$ is not multiplicative.

Manifolds with many parallel spinors

Theorem 3 *Let (M, g) be a pseudo-Riemannian spin manifold.*

(i) If $\kappa(M) > \frac{3}{4}$, then (M, g) is flat.

(ii) If the metric g is definite and $\kappa(M) > \frac{1}{4}$, then (M, g) is flat. A complete simply connected Riemannian spin manifold (M, g) with $\kappa(M) = \frac{1}{4}$ is the product of a flat manifold and a manifold with holonomy group SU(2)*.*

Theorem 4 *Let $(\widehat{M}, \widehat{g})$ be the Lorentzian cone over a pseudo-Riemannian manifold (M, g) of signature* $(0, n)$ *or* $(n-1, 1)$*. If $\kappa(\widehat{M}) > \frac{1}{2}$, then $\widehat{M}$ is flat and M has constant curvature* 1*.*

Cones $\hat{M}$ over pseudo-Riemannian manifolds M and its spinor structure

Definition 4 *Let* (M, g) *be a pseudo-Riemannian manifold of signature* (p, q)*. The manifold* $\hat{M} = \mathbb{R}^+ \times M$ *endowed with the pseudo-Riemannian metric* $\hat{g} = dr^2 + r^2 g$ *of signature* $(p+1, q)$ *is called the cone over* (M, g)*.*

Recall that a spin structure on (M, g) is a $\mathrm{Spin}_0(p,q)$-equivariant two-fold covering
$P_{\mathrm{Spin}_0(p,q)}(M) \to P_{SO_0(p,q)}(M)$
of the principal bundle of orthonormal frames.

Enlarging the structure groups, we define the principal bundles $P_{\mathrm{Spin}_0(p+1,q)}(M)$ and $P_{SO_0(p+1,q)}(M)$ over M.

The inclusion
$M \cong \{1\} \times M \subset \widehat{M} = \mathbb{R}^+ \times M$, defines identifications

$$P_{SO_0(p+1,q)}(M) = P_{SO_0(p+1,q)}(\widehat{M})|_M$$

$$P_{SO_0(p+1,q)}(\widehat{M}) = \pi^*(P_{SO_0(p+1,q)}(M))$$

where $\pi : \widehat{M} \to M$ is the projection.

We can extend the spinor structure of M to a spinor structure on $\widehat{M}$ using:

$$P_{\mathsf{Spin}_0(p+1,q)}(\widehat{M}) := \pi^* P_{\mathsf{Spin}_0(p+1,q)}(M) \to$$

$$\to \pi^* P_{SO_0(p+1,q)}(M) = P_{SO_0(p+1,q)}(\widehat{M}).$$

Here

$$\Theta : P_{\mathsf{Spin}_0(p+1,q)}(M) \to P_{SO_0(p+1,q)}(M)$$

is the natural extension of the covering

$$P_{\mathsf{Spin}_0(p,q)}(M) \to P_{SO_0(p,q)}(M).$$

Relation between spinor bundles of M and the cone $\hat{M}$

Lemma 2 *Let* $(\hat{M}, \hat{g})$ *be the cone over a spin manifold* (M, g) *of signature* (p, q)*.*

(i) If $s = p - q \equiv 0, 2, 4, 5$ *or* $6 \pmod 8$*, then*

$$\hat{S}|_M \cong S.$$

(ii) If $s = p - q \equiv 1, 3$ *or* $7 \pmod 8$*, then*

$$\hat{S}^{\pm}|_M \cong S.$$

(iii) If $n = \dim M = p + q$ *is even,*

$$\hat{\mathbb{S}}|_M \cong \mathbb{S}.$$

where $\mathbb{S}$ *is the complex spinor bundles.*

(iv) If n *is odd, then* $\hat{\mathbb{S}}^{\pm}|_M \cong \mathbb{S}.$

Relation between Killing spinors of M and parallel spinors of the cone $\hat{M}$

Theorem 5 *(Ch. Bär)*
i) The restriction $\Gamma(\widehat{S}) \ni s \mapsto s|_M \in \Gamma(S)$ *defines isomorphisms*

$$\widehat{S}(0) \to S\left(\frac{1}{2}\right) \cong S\left(-\frac{1}{2}\right),$$

if $s = p - q \equiv 0, 2, 4, 5$ *or* $6 \pmod 8$ *and*

$$\widehat{S}^{\pm}(0) \to S\left(\pm\frac{\epsilon}{2}\right),$$

for some $\epsilon \in \{1, -1\}$, *if* $s = p - q \equiv 1, 3$ *or* $7 \pmod 8$.
ii) The restriction $\Gamma(\widehat{\mathbb{S}}) \ni s \mapsto s|_M \in \Gamma(\mathbb{S})$ *defines isomorphisms*

$$\widehat{\mathbb{S}}(0) \to \mathbb{S}\left(\frac{1}{2}\right) \cong \mathbb{S}\left(-\frac{1}{2}\right),$$

if $n = \dim M$ *is even and*

$$\widehat{\mathbb{S}}^{\pm}(0) \to \mathbb{S}\left(\pm\frac{\epsilon}{2}\right),$$

for some $\epsilon \in \{1, -1\}$, *if* n *is odd.*

Riemannian manifolds, whose cone has parallel spinors

Theorem 6 *Let (M, g) be a simply connected Riemannian spin manifold and one of the conditions holds:*

a) *(M, g) is complete and not of constant curvature 1.*

b) *The holonomy algebra of M is different from $\mathfrak{so}(n)$, where $n = \dim M$.*

Then the holonomy algebra $\widehat{\mathfrak{h}}$ of the cone $(\widehat{M}, \widehat{g})$ is irreducible. The cone admits a parallel spinor iff $\widehat{\mathfrak{h}} = \mathfrak{su}(m)$ $(m \geq 3, k = 2)$, $\mathfrak{sp}(m)$ $(m \geq 2, k = m + 1)$, $\mathfrak{spin}(7)$ $(k = 1)$ or $\mathfrak{g}_2$ $(k = 1)$ where k in brackets indicates the number of parallel complex spinors.

Recall that according to Ch. Bär the holonomy algebra $\widehat{\mathfrak{h}}$ of the cone $(\widehat{M}, \widehat{g})$ over a simply connected Riemannian manifold (M, g) belongs to the list of irreducible linear Lie algebras described in above Theorem :

$$\mathfrak{su}(m),\ (m \geq 3),\ \mathfrak{sp}(m),\ (m \geq 2),\ \mathfrak{spin}(7),\ \mathfrak{g}_2$$

iff (M, g) is Einstein-Sasaki, 3-Sasakian, strictly nearly parallel G_2 or strictly nearly Kähler, respectively. We will call these geometric structures on (M, g) Bär geometries.

Estimation of $\kappa(U, \lambda)$

Theorem 7 *Let (M, g) be a Riemannian spin manifold which is not of constant positive curvature $4\lambda^2$.*

(i) Then $\kappa(M, \lambda) \leq \frac{3}{8}$.

(ii) If $\kappa(U, \lambda) = \frac{3}{8}$ for small neighborhood $U \subset M$ of any point p. Then either
(M, g) is locally isometric to seven-dimensional 3-Sasakian manifold or
there exists a dense open subset $M' \subset M$ such that every point of M' has a neighborhood isometric to a Riemannian manifold of the form

$$(I \times M_1 \times M_2, ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2),$$

where (M_1, g_1) is of constant curvature 1 or of dimension ≤ 1 and (M_2, g_2) is a seven-dimensional 3-Sasakian manifold.

Structure of a Riemannian spin manifold with parallel spinor

Theorem 8 *Let* (M, g) *be an* n*-dimensional Riemannian spin manifold which admits a non-trivial parallel spinor.*
Then either (M, g)
is locally a product $M = M_0 \times M_1 \times \cdots \times M_r$ *of a flat Riemannian manifold* M_0 *with an arbitrary number of Riemannian manifolds* M_i *with irreducible holonomy group from the following list:* SU(m), Sp(m), Spin(7) *or* G$_2$.

Inductive construction of manifolds with Killing spinor

Theorem 9 *Let (M,g) be an n-dimensional Riemannian spin manifold which admits a nontrivial Killing spinor with Killing constant $\lambda \in \mathbb{R} \setminus \{0\}$. Then (M,g) has holonomy $\mathfrak{h} = \mathfrak{so}(n)$. Moreover, if the cone $\hat{M}$ is locally irreducible, then (M,g) carries locally one of the Bär geometries and if $\hat{M}$ is locally reducible, then, on a dense open subset, (M,g) can be locally represented in the form*

$$M = I \times M_1 \times M_2, \quad g = ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2, \tag{11}$$

where $I \subset (0, \frac{\pi}{2})$ is an intervall and (M_1, g_1) and (M_2, g_2) are Riemannian manifolds which either admit a nontrivial Killing spinor with Killing constant $\pm\lambda$ or which are of dimension ≤ 1.

Pseudo-Riemannian manifolds with Lorentzian cone, which admit many Killing spinors

Theorem 10 *Let (M,g) be spin with either a negative definite metric or a metric of Lorentzian signature $(+,\dots,+,-)$. If (M,g) is not of positive constant curvature $4\lambda^2$, then $\kappa(M,\lambda) \leq \frac{1}{2}$.*

Remark that a pseudo-Riemannian manifold (M,g) of dimension n which admits a Killing spinor with (real) Killing number $\lambda \in \mathbb{R}\backslash\{0\}$ has positive scalar curvature $s = 4n(n-1)\lambda^2$. If g is negative definite of scalar curvature $s > 0$, then the Riemannian metric $-g$ has negative scalar curvature $-s$. This allows to treat also Riemannian manifolds with negative scalar curvature.

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