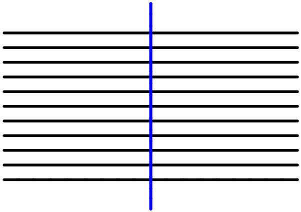
Cohomogeneity One Actions on Symmetric Spaces of Noncompact Type

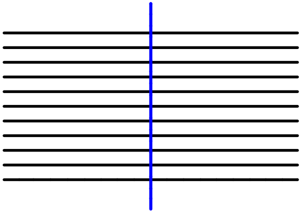
Jürgen Berndt
King's College London

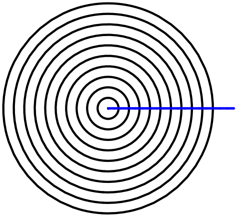
International Summer School on Lorentzian Geometry
Analysis and Geometry of Pseudo-Riemannian Manifolds
Greifswald, 20-22 July 2009

Examples of cohomogeneity one actions

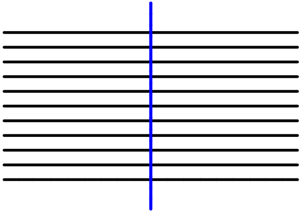


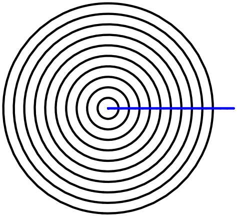
Examples of cohomogeneity one actions

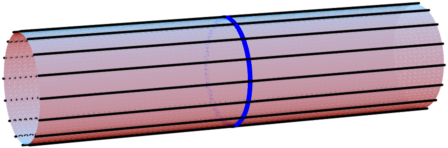




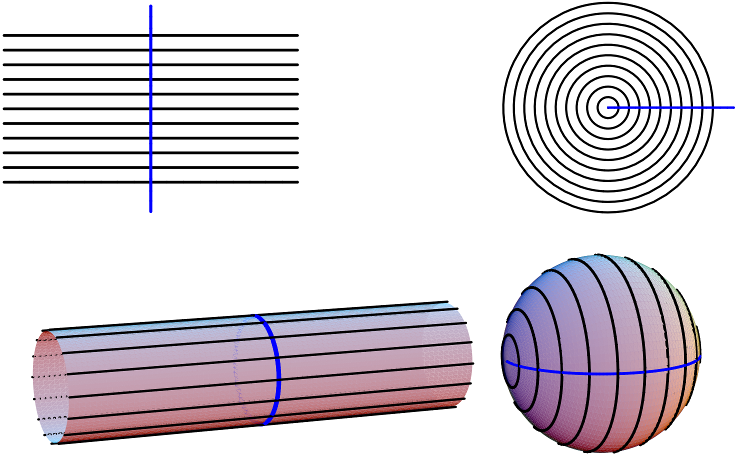
Examples of cohomogeneity one actions







Examples of cohomogeneity one actions



The orbit space of a cohomogeneity one action

- M connected complete Riemannian manifold
  $I(M)$ isometry group of M

The orbit space of a cohomogeneity one action

- M connected complete Riemannian manifold
  $I(M)$ isometry group of M
- $H \subset I(M)$ connected closed subgroup acting on M with cohomogeneity one

The orbit space of a cohomogeneity one action

- M connected complete Riemannian manifold
  $I(M)$ isometry group of M
- $H \subset I(M)$ connected closed subgroup acting on M with cohomogeneity one
- $M/H = \{H \cdot p \mid p \in M\}$ orbit space

The orbit space of a cohomogeneity one action

- M connected complete Riemannian manifold
  $I(M)$ isometry group of M
- $H \subset I(M)$ connected closed subgroup acting on M with cohomogeneity one
- $M/H = \{H \cdot p \mid p \in M\}$ orbit space

$$\boxed{M/H \quad \cong \quad \mathbb{R}\ ,\ S^1\ ,\ [0,\infty) \text{ or } [0,1]}$$

The orbit space of a cohomogeneity one action

- M connected complete Riemannian manifold
  $I(M)$ isometry group of M
- $H \subset I(M)$ connected closed subgroup acting on M with cohomogeneity one
- $M/H = \{H \cdot p \mid p \in M\}$ orbit space

$$\boxed{M/H \quad \cong \quad \mathbb{R}\ ,\ S^1\ ,\ [0,\infty) \text{ or } [0,1]}$$

- $M/H \cong \mathbb{R}, S^1$
  $\Rightarrow$ orbits form a Riemannian foliation on M

The orbit space of a cohomogeneity one action

- M connected complete Riemannian manifold
  $I(M)$ isometry group of M
- $H \subset I(M)$ connected closed subgroup acting on M with cohomogeneity one
- $M/H = \{H \cdot p \mid p \in M\}$ orbit space

$$\boxed{M/H \quad \cong \quad \mathbb{R}\ ,\ S^1\ ,\ [0,\infty) \text{ or } [0,1]}$$

- $M/H \cong \mathbb{R}, S^1$
  $\Rightarrow$ orbits form a Riemannian foliation on M
- $M/H \cong [0,\infty), [0,1]$
  boundary point $\longleftrightarrow$ singular orbit
  interior point $\longleftrightarrow$ principal orbit

Some general results

- $\pi_1(M)$ finite $\Longrightarrow M/H \ncong S^1$

Some general results

- $\pi_1(M)$ finite $\Longrightarrow M/H \ncong S^1$
- $\pi_1(M) = 0$, M compact $\Longrightarrow M/H \cong [0,1]$

Some general results

- $\pi_1(M)$ finite $\Longrightarrow M/H \ncong S^1$
- $\pi_1(M) = 0$, M compact $\Longrightarrow M/H \cong [0, 1]$
- $\pi_1(M) = 0$, $sect_M \leq 0 \Longrightarrow M/H \cong \mathbb{R}$ or $[0, \infty)$

Cohomogeneity one method

- M connected compact smooth manifold

Cohomogeneity one method

- M connected compact smooth manifold
- G compact Lie group acting on M with cohomogeneity one

Cohomogeneity one method

- M connected compact smooth manifold
- G compact Lie group acting on M with cohomogeneity one
- Assume $M/G \cong [0,1]$, M_0, M_1 singular orbits

Cohomogeneity one method

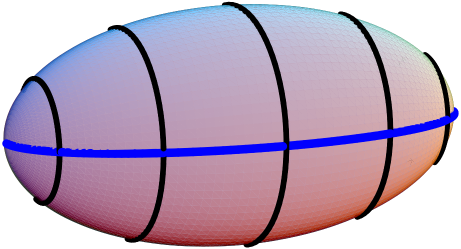
- M connected compact smooth manifold
- G compact Lie group acting on M with cohomogeneity one
- Assume $M/G \cong [0,1]$, M_0, M_1 singular orbits
- $\tilde{M} = M \setminus (M_0 \cup M_1) =$ union of principal orbits
  $\tilde{M} \overset{\mathrm{diff}}{=} (0,1) \times P$ with P a principal orbit

Cohomogeneity one method

- M connected compact smooth manifold
- G compact Lie group acting on M with cohomogeneity one
- Assume $M/G \cong [0,1]$, M_0, M_1 singular orbits
- $\tilde{M} = M \setminus (M_0 \cup M_1) =$ union of principal orbits
  $\tilde{M} \overset{\mathrm{diff}}{=} (0,1) \times P$ with P a principal orbit
- $g = dt^2 + g_t \ \text{ on } \tilde{M}$
  g_t one-parameter family of G-invariant metrics on P

Cohomogeneity one method

- M connected compact smooth manifold
- G compact Lie group acting on M with cohomogeneity one
- Assume $M/G \cong [0,1]$, M_0, M_1 singular orbits
- $\tilde{M} = M \setminus (M_0 \cup M_1) =$ union of principal orbits
  $\tilde{M} \overset{\text{diff}}{=} (0,1) \times P$ with P a principal orbit
- $g = dt^2 + g_t \;\; \text{on} \; \tilde{M}$
  g_t one-parameter family of G-invariant metrics on P



Applications

- metrics of holonomy G_2 or $Spin_7$ (**Gibbons**, **Pope**, ...)

Applications

- metrics of holonomy G_2 or $Spin_7$ (**Gibbons**, **Pope**, ...)
- hyperkähler and quaternionic Kähler structures (**Dancer**, **Swann**, ...)

Applications

- metrics of holonomy G_2 or $Spin_7$ (**Gibbons**, **Pope**, ...)
- hyperkähler and quaternionic Kähler structures (**Dancer**, **Swann**, ...)
- Einstein, Einstein-Kähler and Einstein-Weyl structures (**Berard-Bergery**, **Bonneau**, **Dancer**, **Wang**...)

Applications

- metrics of holonomy G_2 or $Spin_7$ (**Gibbons**, **Pope**, ...)
- hyperkähler and quaternionic Kähler structures (**Dancer**, **Swann**, ...)
- Einstein, Einstein-Kähler and Einstein-Weyl structures (**Berard-Bergery**, **Bonneau**, **Dancer**, **Wang**...)
- metrics with positive or nonnegative Ricci or sectional curvature (**Grove**, **Wilking**, **Ziller**, ...)

Applications

- metrics of holonomy G_2 or $Spin_7$ (**Gibbons**, **Pope**, ...)
- hyperkähler and quaternionic Kähler structures (**Dancer**, **Swann**, ...)
- Einstein, Einstein-Kähler and Einstein-Weyl structures (**Berard-Bergery**, **Bonneau**, **Dancer**, **Wang**...)
- metrics with positive or nonnegative Ricci or sectional curvature (**Grove**, **Wilking**, **Ziller**, ...)
- harmonic maps, Yang-Mills equations (**Urakawa**, ...)

Applications

- metrics of holonomy G_2 or $Spin_7$ (**Gibbons**, **Pope**, ...)
- hyperkähler and quaternionic Kähler structures (**Dancer**, **Swann**, ...)
- Einstein, Einstein-Kähler and Einstein-Weyl structures (**Berard-Bergery**, **Bonneau**, **Dancer**, **Wang**...)
- metrics with positive or nonnegative Ricci or sectional curvature (**Grove**, **Wilking**, **Ziller**, ...)
- harmonic maps, Yang-Mills equations (**Urakawa**, ...)
- special Lagrangian submanifolds (**Joyce**, **Min-Oo**, ...)

Applications

- metrics of holonomy G_2 or $Spin_7$ (**Gibbons**, **Pope**, ...)
- hyperkähler and quaternionic Kähler structures (**Dancer**, **Swann**, ...)
- Einstein, Einstein-Kähler and Einstein-Weyl structures (**Berard-Bergery**, **Bonneau**, **Dancer**, **Wang**...)
- metrics with positive or nonnegative Ricci or sectional curvature (**Grove**, **Wilking**, **Ziller**, ...)
- harmonic maps, Yang-Mills equations (**Urakawa**, ...)
- special Lagrangian submanifolds (**Joyce**, **Min-Oo**, ...)
- differential topology (**Atiyah**, **Berndt**, ...)

Spheres

Hsiang-Lawson 1971: *Every cohomogeneity one action on a sphere is orbit equivalent to the isotropy representation of a semisimple Riemannian symmetric space of rank* 2

Spheres

Hsiang-Lawson 1971: *Every cohomogeneity one action on a sphere is orbit equivalent to the isotropy representation of a semisimple Riemannian symmetric space of rank* 2

compact simple Riemannian symmetric spaces of rank 2:

$$SU_3/SO_3\ ,\ SU_3\ ,\ SU_6/Sp_3\ ,\ E_6/F_4$$

$$Sp_2\ ,\ SO_{10}/U_5\ ,\ E_6/Spin_{10}U_1$$

$$SO_{n+2}/SO_nSO_2\ ,\ SU_{n+2}/S(U_nU_2)\ ,\ Sp_{n+2}/Sp_nSp_2$$

$$G_2/SO_4\ ,\ G_2$$

Known classifications

Compact spaces:

Known classifications

Compact spaces:

- **Takagi 1973**: $\mathbb{C}P^n$

Known classifications

Compact spaces:

- **Takagi 1973**: $\mathbb{C}P^n$
- **Uchida 1977**: $H^*(M, \mathbb{Q}) = H^*(\mathbb{C}P^n, \mathbb{Q})$

Known classifications

Compact spaces:

- **Takagi 1973**: $\mathbb{C}P^n$
- **Uchida 1977**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{C}P^n,\mathbb{Q})$
- **Iwata 1978**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{H}P^n,\mathbb{Q})$

Known classifications

Compact spaces:

- **Takagi 1973**: $\mathbb{C}P^n$
- **Uchida 1977**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{C}P^n,\mathbb{Q})$
- **Iwata 1978**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{H}P^n,\mathbb{Q})$
- **Iwata 1981**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{O}P^2,\mathbb{Q})$

Known classifications

Compact spaces:

- **Takagi 1973**: $\mathbb{C}P^n$
- **Uchida 1977**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{C}P^n,\mathbb{Q})$
- **Iwata 1978**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{H}P^n,\mathbb{Q})$
- **Iwata 1981**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{O}P^2,\mathbb{Q})$
- **Kollross 2001**: simply connected irreducible Riemannian symmetric spaces of compact type and rank ≥ 2

Known classifications

Compact spaces:

- **Takagi 1973**: $\mathbb{C}P^n$
- **Uchida 1977**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{C}P^n,\mathbb{Q})$
- **Iwata 1978**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{H}P^n,\mathbb{Q})$
- **Iwata 1981**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{O}P^2,\mathbb{Q})$
- **Kollross 2001**: simply connected irreducible Riemannian symmetric spaces of compact type and rank ≥ 2

Noncompact spaces:

- **Segre 1938**: $\mathbb{E}^n$

Known classifications

Compact spaces:

- **Takagi 1973**: $\mathbb{C}P^n$
- **Uchida 1977**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{C}P^n,\mathbb{Q})$
- **Iwata 1978**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{H}P^n,\mathbb{Q})$
- **Iwata 1981**: $H^*(M,\mathbb{Q}) = H^*(\mathbb{O}P^2,\mathbb{Q})$
- **Kollross 2001**: simply connected irreducible Riemannian symmetric spaces of compact type and rank ≥ 2

Noncompact spaces:

- **Segre 1938**: $\mathbb{E}^n$
- **Cartan 1938**: $\mathbb{R}H^n$

Euclidean spaces

$I^o(\mathbb{E}^n) = \mathbb{E}^n \rtimes SO_n$

Euclidean spaces

$I^o(\mathbb{E}^n) = \mathbb{E}^n \rtimes SO_n$

Segre 1938: *Every cohomogeneity one action on* $\mathbb{E}^n$ *is orbit equivalent to one of the following:*

Euclidean spaces

$I^o(\mathbb{E}^n) = \mathbb{E}^n \rtimes SO_n$

Segre 1938: *Every cohomogeneity one action on $\mathbb{E}^n$ is orbit equivalent to one of the following:*

- $H = \mathbb{E}^{n-1}$*: orbits give a foliation by parallel hyperplanes*

Euclidean spaces

$I^o(\mathbb{E}^n) = \mathbb{E}^n \rtimes SO_n$

Segre 1938: *Every cohomogeneity one action on $\mathbb{E}^n$ is orbit equivalent to one of the following:*

- $H = \mathbb{E}^{n-1}$*: orbits give a foliation by parallel hyperplanes*
- $H = \mathbb{E}^k \rtimes SO_{n-k}$, $0 \le k < n-1$*: orbits are a totally geodesic* $\mathbb{E}^k$ *and the tubes around it*

Euclidean spaces

$I^o(\mathbb{E}^n) = \mathbb{E}^n \rtimes SO_n$

Segre 1938: *Every cohomogeneity one action on $\mathbb{E}^n$ is orbit equivalent to one of the following:*

- $H = \mathbb{E}^{n-1}$: *orbits give a foliation by parallel hyperplanes*
- $H = \mathbb{E}^k \rtimes SO_{n-k}$, $0 \leq k < n-1$: *orbits are a totally geodesic $\mathbb{E}^k$ and the tubes around it*

Proof: use Gauss-Codazzi equations

Real hyperbolic spaces

$I^o(\mathbb{R}H^n) = SO^o_{1,n}$

Real hyperbolic spaces

$I^o(\mathbb{R}H^n) = SO^o_{1,n}$

Cartan 1938: *Every cohomogeneity one action on $\mathbb{R}H^n$ is orbit equivalent to one of the following:*

Real hyperbolic spaces

$I^o(\mathbb{R}H^n) = SO^o_{1,n}$

Cartan 1938: *Every cohomogeneity one action on $\mathbb{R}H^n$ is orbit equivalent to one of the following:*

- $H = SO^o_{1,n-1}$*: orbits give a foliation by a totally geodesic hyperplane* $\mathbb{R}H^{n-1}$ *and its equidistant hypersurfaces*

Real hyperbolic spaces

$I^o(\mathbb{R}H^n) = SO^o_{1,n}$

Cartan 1938: *Every cohomogeneity one action on $\mathbb{R}H^n$ is orbit equivalent to one of the following:*

- $H = SO^o_{1,n-1}$: *orbits give a foliation by a totally geodesic hyperplane $\mathbb{R}H^{n-1}$ and its equidistant hypersurfaces*
- $H = SO^o_{1,k} \times SO_{n-k}$, $0 \leq k < n-1$: *orbits are a totally geodesic $\mathbb{R}H^k$ and the tubes around it*

Real hyperbolic spaces

$I^o(\mathbb{R}H^n) = SO^o_{1,n}$

Cartan 1938: *Every cohomogeneity one action on $\mathbb{R}H^n$ is orbit equivalent to one of the following:*

- $H = SO^o_{1,n-1}$: *orbits give a foliation by a totally geodesic hyperplane $\mathbb{R}H^{n-1}$ and its equidistant hypersurfaces*
- $H = SO^o_{1,k} \times SO_{n-k}$, $0 \leq k < n-1$: *orbits are a totally geodesic $\mathbb{R}H^k$ and the tubes around it*
- $H = N \subset KAN = SO^o_{1,n}$: *orbits give a foliation by horospheres*

Real hyperbolic spaces

$I^o(\mathbb{R}H^n) = SO^o_{1,n}$

Cartan 1938: *Every cohomogeneity one action on $\mathbb{R}H^n$ is orbit equivalent to one of the following:*

- $H = SO^o_{1,n-1}$: *orbits give a foliation by a totally geodesic hyperplane $\mathbb{R}H^{n-1}$ and its equidistant hypersurfaces*
- $H = SO^o_{1,k} \times SO_{n-k}$, $0 \leq k < n-1$: *orbits are a totally geodesic $\mathbb{R}H^k$ and the tubes around it*
- $H = N \subset KAN = SO^o_{1,n}$: *orbits give a foliation by horospheres*

Proof: use Gauss-Codazzi equations

Noncompact symmetric spaces

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
  G noncompact semisimple real Lie group
  K maximal compact subgroup of G
  $o \in M$ with $K \cdot o = o$

Noncompact symmetric spaces

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
  G noncompact semisimple real Lie group
  K maximal compact subgroup of G
  $o \in M$ with $K \cdot o = o$
- H connected closed subgroup of G acting on M with cohomogeneity one

Noncompact symmetric spaces

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
  G noncompact semisimple real Lie group
  K maximal compact subgroup of G
  $o \in M$ with $K \cdot o = o$
- H connected closed subgroup of G acting on M with cohomogeneity one
- L connected proper maximal subgroup of G with $H \subset L$

Noncompact symmetric spaces

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
  G noncompact semisimple real Lie group
  K maximal compact subgroup of G
  $o \in M$ with $K \cdot o = o$
- H connected closed subgroup of G acting on M with cohomogeneity one
- L connected proper maximal subgroup of G with $H \subset L$
- $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{l}$ corresponding Lie algebras

Noncompact symmetric spaces

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
  G noncompact semisimple real Lie group
  K maximal compact subgroup of G
  $o \in M$ with $K \cdot o = o$
- H connected closed subgroup of G acting on M with cohomogeneity one
- L connected proper maximal subgroup of G with $H \subset L$
- $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{l}$ corresponding Lie algebras

Mostow 1961: $\mathfrak{l}$ *is reductive or parabolic*

The reductive case (I)

Karpelevich 1953: *L has a totally geodesic orbit* $\mathcal{S} \subsetneq M$

The reductive case (I)

Karpelevich 1953: *L has a totally geodesic orbit $\mathcal{S} \subsetneq M$*

$$\Longrightarrow H \text{ has a totally geodesic orbit } \mathcal{S}$$

The reductive case (I)

Karpelevich 1953: *L has a totally geodesic orbit $\mathcal{S} \subsetneq M$*

$$\Longrightarrow H \text{ has a totally geodesic orbit } \mathcal{S}$$

$\mathcal{S}$ **reflective**

The reductive case (I)

Karpelevich 1953: *L has a totally geodesic orbit $\mathcal{S} \subsetneq M$*

$$\Longrightarrow H \text{ has a totally geodesic orbit } \mathcal{S}$$

$\mathcal{S}$ **reflective**

$\Updownarrow$

geodesic reflection of M in $\mathcal{S}$ is an isometry

The reductive case (I)

Karpelevich 1953: *L has a totally geodesic orbit $\mathcal{S} \subsetneq M$*

$$\Longrightarrow H \text{ has a totally geodesic orbit } \mathcal{S}$$

$\mathcal{S}$ **reflective**

$\Updownarrow$

geodesic reflection of M in $\mathcal{S}$ is an isometry

$\Updownarrow$

$\exists$ totally geodesic submanifold $\mathcal{S}^\perp$ of M
with $o \in \mathcal{S}^\perp$ and $T_o\mathcal{S}^\perp = \nu_o\mathcal{S}$

The reductive case (I)

Karpelevich 1953: *L has a totally geodesic orbit* $\mathcal{S} \subsetneq M$

$$\Longrightarrow H \text{ has a totally geodesic orbit } \mathcal{S}$$

$$\mathcal{S} \textbf{ reflective}$$
$$\Updownarrow$$
$$\text{geodesic reflection of } M \text{ in } \mathcal{S} \text{ is an isometry}$$
$$\Updownarrow$$
$$\exists \text{ totally geodesic submanifold } \mathcal{S}^\perp \text{ of } M$$
$$\text{with } o \in \mathcal{S}^\perp \text{ and } T_o\mathcal{S}^\perp = \nu_o\mathcal{S}$$

Leung 1979: Classification of reflective submanifolds in irreducible simply connected Riemannian symmetric spaces

The reductive case (II)

Berndt-Tamaru 2004: $\mathcal{S}$ *is a totally geodesic singular orbit of a cohomogeneity one action on* $M \iff$

The reductive case (II)

Berndt-Tamaru 2004: *$\mathcal{S}$ is a totally geodesic singular orbit of a cohomogeneity one action on M* $\iff$

- *$\mathcal{S}$ reflective and* $\operatorname{rank}\mathcal{S}^{\perp} = 1$, *or*

The reductive case (II)

Berndt-Tamaru 2004: *$\mathcal{S}$ is a totally geodesic singular orbit of a cohomogeneity one action on M $\Longleftrightarrow$*

- *$\mathcal{S}$ reflective and $\operatorname{rank}\mathcal{S}^\perp = 1$, or*
- *$\mathcal{S}$ is one of the following totally geodesic non-reflective submanifolds:*

$\mathcal{S}$	M	$\dim S$	$\dim M$
$\mathbb{C}H^2$	G_2^2/SO_4	4	8
$SL_3(\mathbb{R})/SO_3$	G_2^2/SO_4	5	8
G_2^2/SO_4	$SO^o_{3,4}/SO_3SO_4$	8	12
$SL_3(\mathbb{C})/SU_3$	$G_2^{\mathbb{C}}/G_2$	8	14
$G_2^{\mathbb{C}}/G_2$	$SO_7^{\mathbb{C}}/SO_7$	14	21

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

- Λ set of simple roots for Σ

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

- Λ set of simple roots for Σ
- Φ subset of Λ, $\Sigma_\Phi = \Sigma \cap \mathrm{span}\{\Phi\}$

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

- Λ set of simple roots for Σ
- Φ subset of Λ, $\Sigma_\Phi = \Sigma \cap \mathrm{span}\{\Phi\}$
- $\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right)$, $\mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\alpha$
  $\mathfrak{l}_\Phi$ reductive subalgebra, $\mathfrak{n}_\Phi$ nilpotent subalgebra

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

- Λ set of simple roots for Σ
- Φ subset of Λ, $\Sigma_\Phi = \Sigma \cap \mathrm{span}\{\Phi\}$
- $\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right)$, $\mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\alpha$
  $\mathfrak{l}_\Phi$ reductive subalgebra, $\mathfrak{n}_\Phi$ nilpotent subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ parabolic subalgebra; $[\mathfrak{q}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$
  (**Chevalley decomposition**)

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

- Λ set of simple roots for Σ
- Φ subset of Λ, $\Sigma_\Phi = \Sigma \cap \mathrm{span}\{\Phi\}$
- $\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right)$, $\mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\alpha$
  $\mathfrak{l}_\Phi$ reductive subalgebra, $\mathfrak{n}_\Phi$ nilpotent subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ parabolic subalgebra; $[\mathfrak{q}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$ (**Chevalley decomposition**)
- Every parabolic subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{q}_\Phi$ for some subset $\Phi \subset \Lambda$

Parabolic subalgebras (II)

- $\mathfrak{a}_\Phi = \bigcap_{\alpha\in\Phi} \ker\alpha$, $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$
  $\mathfrak{m}_\Phi$ reductive subalgebra, $\mathfrak{a}_\Phi$ abelian subalgebra

Parabolic subalgebras (II)

- $\mathfrak{a}_\Phi = \cap_{\alpha\in\Phi} \ker\alpha$, $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$
  $\mathfrak{m}_\Phi$ reductive subalgebra, $\mathfrak{a}_\Phi$ abelian subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ (**Langlands decomposition**)
  $[\mathfrak{q}_\Phi, \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi] \subset \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$

Parabolic subalgebras (II)

- $\mathfrak{a}_\Phi = \cap_{\alpha\in\Phi}\ker\alpha$, $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$
  $\mathfrak{m}_\Phi$ reductive subalgebra, $\mathfrak{a}_\Phi$ abelian subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ (**Langlands decomposition**)
  $[\mathfrak{q}_\Phi, \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi] \subset \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$
- $\mathfrak{k}_\Phi = \mathfrak{k} \cap \mathfrak{q}_\Phi = \mathfrak{k} \cap \mathfrak{m}_\Phi$
  $[\mathfrak{k}_\Phi, \mathfrak{m}_\Phi] \subset \mathfrak{m}_\Phi$, $[\mathfrak{k}_\Phi, \mathfrak{a}_\Phi] = \{0\}$, $[\mathfrak{k}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$

Parabolic subalgebras (II)

- $\mathfrak{a}_\Phi = \cap_{\alpha\in\Phi} \ker\alpha$, $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$
  $\mathfrak{m}_\Phi$ reductive subalgebra, $\mathfrak{a}_\Phi$ abelian subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ (**Langlands decomposition**)
  $[\mathfrak{q}_\Phi, \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi] \subset \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$
- $\mathfrak{k}_\Phi = \mathfrak{k} \cap \mathfrak{q}_\Phi = \mathfrak{k} \cap \mathfrak{m}_\Phi$
  $[\mathfrak{k}_\Phi, \mathfrak{m}_\Phi] \subset \mathfrak{m}_\Phi$, $[\mathfrak{k}_\Phi, \mathfrak{a}_\Phi] = \{0\}$, $[\mathfrak{k}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$
- $F_\Phi^s = M_\Phi \cdot o$ semisimple symmetric space with rank equal to $|\Phi|$, totally geodesic in M, boundary component of M w.r.t. maximal Satake compactification

Parabolic subalgebras (II)

- $\mathfrak{a}_\Phi = \cap_{\alpha\in\Phi} \ker \alpha$, $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$
  $\mathfrak{m}_\Phi$ reductive subalgebra, $\mathfrak{a}_\Phi$ abelian subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ (**Langlands decomposition**)
  $[\mathfrak{q}_\Phi, \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi] \subset \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$
- $\mathfrak{k}_\Phi = \mathfrak{k} \cap \mathfrak{q}_\Phi = \mathfrak{k} \cap \mathfrak{m}_\Phi$
  $[\mathfrak{k}_\Phi, \mathfrak{m}_\Phi] \subset \mathfrak{m}_\Phi$, $[\mathfrak{k}_\Phi, \mathfrak{a}_\Phi] = \{0\}$, $[\mathfrak{k}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$
- $F^s_\Phi = M_\Phi \cdot o$ semisimple symmetric space with rank equal to $|\Phi|$, totally geodesic in M, boundary component of M w.r.t. maximal Satake compactification
- $A_\Phi \cdot o = \mathbb{E}^{r-|\Phi|}$ Euclidean space, totally geodesic in M

Parabolic subalgebras (II)

- $\mathfrak{a}_\Phi = \cap_{\alpha\in\Phi} \ker\alpha$, $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$
  $\mathfrak{m}_\Phi$ reductive subalgebra, $\mathfrak{a}_\Phi$ abelian subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ (**Langlands decomposition**)
  $[\mathfrak{q}_\Phi, \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi] \subset \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$
- $\mathfrak{k}_\Phi = \mathfrak{k} \cap \mathfrak{q}_\Phi = \mathfrak{k} \cap \mathfrak{m}_\Phi$
  $[\mathfrak{k}_\Phi, \mathfrak{m}_\Phi] \subset \mathfrak{m}_\Phi$, $[\mathfrak{k}_\Phi, \mathfrak{a}_\Phi] = \{0\}$, $[\mathfrak{k}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$
- $F^s_\Phi = M_\Phi \cdot o$ semisimple symmetric space with rank equal to $|\Phi|$, totally geodesic in M, boundary component of M w.r.t. maximal Satake compactification
- $A_\Phi \cdot o = \mathbb{E}^{r-|\Phi|}$ Euclidean space, totally geodesic in M
- $L_\Phi \cdot o = F_\Phi = F^s_\Phi \times \mathbb{E}^{r-|\Phi|}$ totally geodesic in M

Parabolic subalgebras (II)

- $\mathfrak{a}_\Phi = \cap_{\alpha\in\Phi} \ker\alpha$, $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$
  $\mathfrak{m}_\Phi$ reductive subalgebra, $\mathfrak{a}_\Phi$ abelian subalgebra
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ (**Langlands decomposition**)
  $[\mathfrak{q}_\Phi, \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi] \subset \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$
- $\mathfrak{k}_\Phi = \mathfrak{k} \cap \mathfrak{q}_\Phi = \mathfrak{k} \cap \mathfrak{m}_\Phi$
  $[\mathfrak{k}_\Phi, \mathfrak{m}_\Phi] \subset \mathfrak{m}_\Phi$, $[\mathfrak{k}_\Phi, \mathfrak{a}_\Phi] = \{0\}$, $[\mathfrak{k}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$
- $F^s_\Phi = M_\Phi \cdot o$ semisimple symmetric space with rank equal to $|\Phi|$, totally geodesic in M, boundary component of M w.r.t. maximal Satake compactification
- $A_\Phi \cdot o = \mathbb{E}^{r-|\Phi|}$ Euclidean space, totally geodesic in M
- $L_\Phi \cdot o = F_\Phi = F^s_\Phi \times \mathbb{E}^{r-|\Phi|}$ totally geodesic in M
- $M = F^s_\Phi \times \mathbb{E}^{r-|\Phi|} \times N_\Phi$ (**horospherical decomposition**)

The boundary components

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$

The boundary components

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i \in \mathfrak{a} \subset \mathfrak{p} \cong T_o M$

The boundary components

- $\Lambda = \{\alpha_1, \dots, \alpha_r\}$, $\{H^1, \dots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i \in \mathfrak{a} \subset \mathfrak{p} \cong T_o M$
- γ^Φ geodesic in M tangent to H^Φ

The boundary components

- $\Lambda = \{\alpha_1, \dots, \alpha_r\}$, $\{H^1, \dots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i \in \mathfrak{a} \subset \mathfrak{p} \cong T_o M$
- γ^Φ geodesic in M tangent to H^Φ
- $F_\Phi = \{p \in M \mid p \text{ lies on geodesic parallel to } \gamma^\Phi\}$

The boundary components

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i \in \mathfrak{a} \subset \mathfrak{p} \cong T_o M$
- γ^Φ geodesic in M tangent to H^Φ
- $F_\Phi = \{p \in M \mid p \text{ lies on geodesic parallel to } \gamma^\Phi\}$
- F_Φ^s = semisimple part of F_Φ

The boundary components

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i \in \mathfrak{a} \subset \mathfrak{p} \cong T_o M$
- γ^Φ geodesic in M tangent to H^Φ
- $F_\Phi = \{p \in M \mid p \text{ lies on geodesic parallel to } \gamma^\Phi\}$
- $F_\Phi^s =$ semisimple part of F_Φ
- $\mathbb{E}^{r-|\Phi|} =$ Euclidean part of F_Φ

Construction Method I: $\nu_o(H \cdot o) \subset \mathfrak{m}_\Phi$, $\Phi \neq \emptyset$

- $H^s_\Phi \subset I^o(F^s_\Phi) \subset M_\Phi$ acting on F^s_Φ with cohomogeneity one

Construction Method I: $\nu_o(H \cdot o) \subset \mathfrak{m}_\Phi$, $\Phi \neq \emptyset$

- $H^s_\Phi \subset I^o(F^s_\Phi) \subset M_\Phi$ acting on F^s_Φ with cohomogeneity one
- $\mathfrak{h} = \mathfrak{h}^s_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ subalgebra of $\mathfrak{q}_\Phi$

Construction Method I: $\nu_o(H \cdot o) \subset \mathfrak{m}_\Phi$, $\Phi \neq \emptyset$

- $H_\Phi^s \subset I^o(F_\Phi^s) \subset M_\Phi$ acting on F_Φ^s with cohomogeneity one
- $\mathfrak{h} = \mathfrak{h}_\Phi^s \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ subalgebra of $\mathfrak{q}_\Phi$

H acts on M with cohomogeneity one

Construction Method I: $\nu_o(H \cdot o) \subset \mathfrak{m}_\Phi$, $\Phi \neq \emptyset$

- $H^s_\Phi \subset I^o(F^s_\Phi) \subset M_\Phi$ acting on F^s_Φ with cohomogeneity one
- $\mathfrak{h} = \mathfrak{h}^s_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ subalgebra of $\mathfrak{q}_\Phi$

H acts on M with cohomogeneity one

Rank reduction - Such a cohomogeneity one action can be constructed by a CANONICAL EXTENSION OF A COHOMOGENEITY ONE ACTION ON A BOUNDARY COMPONENT

Construction Method II: $\nu_o(H \cdot o) \subset \mathfrak{a}_\Phi$

- $\Phi = \emptyset$: $\mathfrak{q}_\Phi = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra

Construction Method II: $\nu_o(H \cdot o) \subset \mathfrak{a}_\Phi$

- $\Phi = \emptyset$: $\mathfrak{q}_\Phi = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra
- $M = AN$ solvable Lie group with left-invariant metric

Construction Method II: $\nu_o(H \cdot o) \subset \mathfrak{a}_\Phi$

- $\Phi = \emptyset$: $\mathfrak{q}_\Phi = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra
- $M = AN$ solvable Lie group with left-invariant metric
- $\ell \subset \mathfrak{a}$ one-dimensional linear subspace

Construction Method II: $\nu_o(H \cdot o) \subset \mathfrak{a}_\Phi$

- $\Phi = \emptyset$: $\mathfrak{q}_\Phi = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra
- $M = AN$ solvable Lie group with left-invariant metric
- $\ell \subset \mathfrak{a}$ one-dimensional linear subspace
- $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$

Construction Method II: $\nu_o(H \cdot o) \subset \mathfrak{a}_\Phi$

- $\Phi = \emptyset$: $\mathfrak{q}_\Phi = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra
- $M = AN$ solvable Lie group with left-invariant metric
- $\ell \subset \mathfrak{a}$ one-dimensional linear subspace
- $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$

S_ℓ acts on M with cohomogeneity one

Construction Method II: $\nu_o(H \cdot o) \subset \mathfrak{a}_\Phi$

- $\Phi = \emptyset$: $\mathfrak{q}_\Phi = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra
- $M = AN$ solvable Lie group with left-invariant metric
- $\ell \subset \mathfrak{a}$ one-dimensional linear subspace
- $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$

S_ℓ acts on M with cohomogeneity one

Construction method II produces abelian extensions of horocycle foliations on M and includes all HOROSPHERE FOLIATIONS on M

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Assume that

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \dots, \alpha_r\}$, $\{H^1, \dots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Assume that

- $\mathfrak{v} \subset \mathfrak{n}_j^1$; define $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ subalgebra of $\mathfrak{n}_j$

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Assume that

- $\mathfrak{v} \subset \mathfrak{n}_j^1$; define $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ subalgebra of $\mathfrak{n}_j$
- $N_{L_j}^o(\mathfrak{n}_{j,\mathfrak{v}}) = \theta N_{L_j}^o(\mathfrak{v})$ acts transitively on $F_j = F_j^s \times \mathbb{E}$

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \dots, \alpha_r\}$, $\{H^1, \dots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Assume that

- $\mathfrak{v} \subset \mathfrak{n}_j^1$; define $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ subalgebra of $\mathfrak{n}_j$
- $N^o_{L_j}(\mathfrak{n}_{j,\mathfrak{v}}) = \theta N^o_{L_j}(\mathfrak{v})$ acts transitively on $F_j = F_j^s \times \mathbb{E}$
- $N^o_{L_j \cap K}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$ if $\dim \mathfrak{v} \geq 2$

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Assume that

- $\mathfrak{v} \subset \mathfrak{n}_j^1$; define $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ subalgebra of $\mathfrak{n}_j$
- $N_{L_j}^o(\mathfrak{n}_{j,\mathfrak{v}}) = \theta N_{L_j}^o(\mathfrak{v})$ acts transitively on $F_j = F_j^s \times \mathbb{E}$
- $N_{L_j \cap K}^o(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$ if $\dim \mathfrak{v} \geq 2$

Then

$$H_{j,\mathfrak{v}} = N_{L_j}^o(\mathfrak{n}_{j,\mathfrak{v}}) N_{j,\mathfrak{v}} \text{ acts on } M \text{ with cohomogeneity one}$$

Construction Method III: $\nu_o(H \cdot o) \subset \mathfrak{n}_\Phi$

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Assume that

- $\mathfrak{v} \subset \mathfrak{n}_j^1$; define $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ subalgebra of $\mathfrak{n}_j$
- $N_{L_j}^o(\mathfrak{n}_{j,\mathfrak{v}}) = \theta N_{L_j}^o(\mathfrak{v})$ acts transitively on $F_j = F_j^s \times \mathbb{E}$
- $N_{L_j \cap K}^o(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$ if $\dim \mathfrak{v} \geq 2$

Then

$$H_{j,\mathfrak{v}} = N_{L_j}^o(\mathfrak{n}_{j,\mathfrak{v}}) N_{j,\mathfrak{v}} \text{ acts on } M \text{ with cohomogeneity one}$$

Remark: $\dim \mathfrak{v} = 1$ corresponds to foliation

Construction Method III - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	$\mathbb{O}$	$\mathbb{O} \oplus \mathbb{R}^7$

Construction Method III - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	$\mathbb{O}$	$\mathbb{O} \oplus \mathbb{R}^7$

- $\Lambda = \{\alpha\}$, $\Phi = \emptyset$, $\mathfrak{l}_\Phi = \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, $\mathfrak{n}_\Phi = \mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$

Construction Method III - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	$\mathbb{O}$	$\mathbb{O} \oplus \mathbb{R}^7$

- $\Lambda = \{\alpha\}$, $\Phi = \emptyset$, $\mathfrak{l}_\Phi = \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, $\mathfrak{n}_\Phi = \mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$
- $\mathfrak{q}_\Phi = \mathfrak{g}_0 \oplus \mathfrak{n} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra

Construction Method III - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	$\mathbb{O}$	$\mathbb{O} \oplus \mathbb{R}^7$

- $\Lambda = \{\alpha\}$, $\Phi = \emptyset$, $\mathfrak{l}_\Phi = \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, $\mathfrak{n}_\Phi = \mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$
- $\mathfrak{q}_\Phi = \mathfrak{g}_0 \oplus \mathfrak{n} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra

<u>Problem:</u> Find all k-dimensional ($k \geq 2$) linear subspaces $\mathfrak{v}$ of $\mathfrak{g}_\alpha$ for which there exists a subgroup of K_0 acting transitively on the unit sphere in $\mathfrak{v}$

Construction Method III - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	$\mathbb{O}$	$\mathbb{O} \oplus \mathbb{R}^7$

Berndt-Tamaru 2007:

- $\mathbb{R}$: any linear subspace $\mathfrak{v} \subset \mathbb{R}^{n-1}$

Construction Method III - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	$\mathbb{O}$	$\mathbb{O} \oplus \mathbb{R}^7$

Berndt-Tamaru 2007:

- $\mathbb{R}$: any linear subspace $\mathfrak{v} \subset \mathbb{R}^{n-1}$
- $\mathbb{C}$: any linear subspace $\mathfrak{v} \subset \mathbb{C}^{n-1}$ with constant Kähler angle

Construction Method III - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	$\mathbb{O}$	$\mathbb{O} \oplus \mathbb{R}^7$

Berndt-Tamaru 2007:

- $\mathbb{R}$: any linear subspace $\mathfrak{v} \subset \mathbb{R}^{n-1}$
- $\mathbb{C}$: any linear subspace $\mathfrak{v} \subset \mathbb{C}^{n-1}$ with constant Kähler angle
- $\mathbb{O}$: any linear subspace $\mathfrak{v} \subset \mathbb{O}$ of dimension $k \in \{2, 3, 4, 6, 7\}$

Construction Method III - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	$\mathbb{O}$	$\mathbb{O} \oplus \mathbb{R}^7$

Berndt-Tamaru 2007:

- $\mathbb{R}$: any linear subspace $\mathfrak{v} \subset \mathbb{R}^{n-1}$
- $\mathbb{C}$: any linear subspace $\mathfrak{v} \subset \mathbb{C}^{n-1}$ with constant Kähler angle
- $\mathbb{O}$: any linear subspace $\mathfrak{v} \subset \mathbb{O}$ of dimension $k \in \{2, 3, 4, 6, 7\}$
- $\mathbb{H}$: some linear subspaces $\mathfrak{v} \subset \mathbb{H}^{n-1}$ with constant quaternionic Kähler angle (no complete classification)

The Classification

Berndt-Tamaru 2008: *Let M be a connected irreducible Riemannian symmetric space of noncompact type.*

The Classification

Berndt-Tamaru 2008: *Let M be a connected irreducible Riemannian symmetric space of noncompact type. Then every cohomogeneity one action on M either has a totally geodesic singular orbit, or it is orbit equivalent to a cohomogeneity one action obtained by construction method I, II or III.*

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (I)

- The boundary components of $SL_3(\mathbb{R})/SO_3$ are $F_1^s \cong F_2^s \cong \mathbb{R}H^2$

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (I)

- The boundary components of $SL_3(\mathbb{R})/SO_3$ are $F_1^s \cong F_2^s \cong \mathbb{R}H^2$
- There are three types of cohomogeneity one actions on $\mathbb{R}H^2 = SL_2(\mathbb{R})/SO_2$. Write $SL_2(\mathbb{R}) = KAN$.

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (I)

- The boundary components of $SL_3(\mathbb{R})/SO_3$ are $F_1^s \cong F_2^s \cong \mathbb{R}H^2$
- There are three types of cohomogeneity one actions on $\mathbb{R}H^2 = SL_2(\mathbb{R})/SO_2$. Write $SL_2(\mathbb{R}) = KAN$.
  - $K = SO_2$: the canonical extension is a cohomogeneity one action with a minimal $\mathbb{R}H^3$ as a singular orbit

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (I)

- The boundary components of $SL_3(\mathbb{R})/SO_3$ are $F_1^s \cong F_2^s \cong \mathbb{R}H^2$
- There are three types of cohomogeneity one actions on $\mathbb{R}H^2 = SL_2(\mathbb{R})/SO_2$. Write $SL_2(\mathbb{R}) = KAN$.
  - $K = SO_2$: the canonical extension is a cohomogeneity one action with a minimal $\mathbb{R}H^3$ as a singular orbit
  - N: the canonical extension leads to a horosphere foliation (with a singular point at infinity)

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (I)

- The boundary components of $SL_3(\mathbb{R})/SO_3$ are $F_1^s \cong F_2^s \cong \mathbb{R}H^2$
- There are three types of cohomogeneity one actions on $\mathbb{R}H^2 = SL_2(\mathbb{R})/SO_2$. Write $SL_2(\mathbb{R}) = KAN$.
  - $K = SO_2$: the canonical extension is a cohomogeneity one action with a minimal $\mathbb{R}H^3$ as a singular orbit
  - N: the canonical extension leads to a horosphere foliation (with a singular point at infinity)
  - A: the canonical extension leads to a foliation with a minimal homogeneous hypersurface P as a leaf, where P is the canonical extension of a geodesic in $\mathbb{R}H^2$

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (I)

- The boundary components of $SL_3(\mathbb{R})/SO_3$ are $F_1^s \cong F_2^s \cong \mathbb{R}H^2$
- There are three types of cohomogeneity one actions on $\mathbb{R}H^2 = SL_2(\mathbb{R})/SO_2$. Write $SL_2(\mathbb{R}) = KAN$.
    - $K = SO_2$: the canonical extension is a cohomogeneity one action with a minimal $\mathbb{R}H^3$ as a singular orbit
    - N: the canonical extension leads to a horosphere foliation (with a singular point at infinity)
    - A: the canonical extension leads to a foliation with a minimal homogeneous hypersurface P as a leaf, where P is the canonical extension of a geodesic in $\mathbb{R}H^2$
- $\mathbb{R}H^2 \times \mathbb{E} \cong F_1 \cong F_2$ is the only reflective submanifold $\mathcal{S}$ in $SL_3(\mathbb{R})/SO_3$ for which $\mathcal{S}^\perp$ has rank one

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (II)

Theorem: *Every cohomogeneity one action on $SL_3(\mathbb{R})/SO_3$ has one of the following orbit structures:*

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (II)

Theorem: *Every cohomogeneity one action on $SL_3(\mathbb{R})/SO_3$ has one of the following orbit structures:*

- *the reflective submanifold $\mathbb{R}H^2 \times \mathbb{E}$ and the tubes around it*

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (II)

Theorem: *Every cohomogeneity one action on $SL_3(\mathbb{R})/SO_3$ has one of the following orbit structures:*

- *the reflective submanifold $\mathbb{R}H^2 \times \mathbb{E}$ and the tubes around it*
- *an abelian extension of a horocycle foliation*

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (II)

Theorem: *Every cohomogeneity one action on $SL_3(\mathbb{R})/SO_3$ has one of the following orbit structures:*

- *the reflective submanifold $\mathbb{R}H^2 \times \mathbb{E}$ and the tubes around it*
- *an abelian extension of a horocycle foliation*
- *the foliation given by the canonical extension of a geodesic in $\mathbb{R}H^2$ to a minimal homogeneous hypersurface, and its equidistant hypersurfaces*

Cohomogeneity one actions on $SL_3(\mathbb{R})/SO_3$ (II)

Theorem: *Every cohomogeneity one action on $SL_3(\mathbb{R})/SO_3$ has one of the following orbit structures:*

- *the reflective submanifold $\mathbb{R}H^2 \times \mathbb{E}$ and the tubes around it*
- *an abelian extension of a horocycle foliation*
- *the foliation given by the canonical extension of a geodesic in $\mathbb{R}H^2$ to a minimal homogeneous hypersurface, and its equidistant hypersurfaces*
- *the minimal $\mathbb{R}H^3$ and the tubes around it*

The case of regular foliations

- $\mathcal{M}_F$ = set of all homogeneous codimension one foliations on M up to isometric congruence

The case of regular foliations

- $\mathcal{M}_F$ = set of all homogeneous codimension one foliations on M up to isometric congruence
- r = rank of M

The case of regular foliations

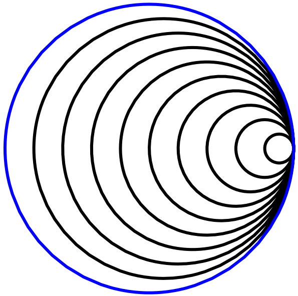
- $\mathcal{M}_F$ = set of all homogeneous codimension one foliations on M up to isometric congruence
- r = rank of M
- $\mathrm{Aut}(DD) \in \{I, \mathbb{Z}_2, \mathfrak{S}_3\}$ automorphism group of the Dynkin diagram associated to M

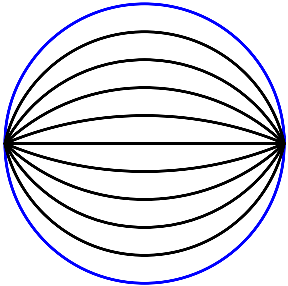
The case of regular foliations

- $\mathcal{M}_F$ = set of all homogeneous codimension one foliations on M up to isometric congruence
- r = rank of M
- $\mathrm{Aut}(DD) \in \{I, \mathbb{Z}_2, \mathfrak{S}_3\}$ automorphism group of the Dynkin diagram associated to M

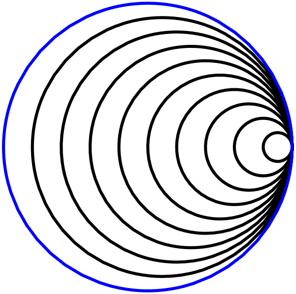
Berndt-Tamaru 2003: $\boxed{\mathcal{M}_F \cong (\mathbb{R}P^{r-1} \cup \{1, \ldots, r\})/\mathrm{Aut}(DD)}$

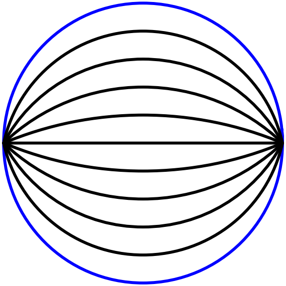
The two foliations on hyperbolic spaces





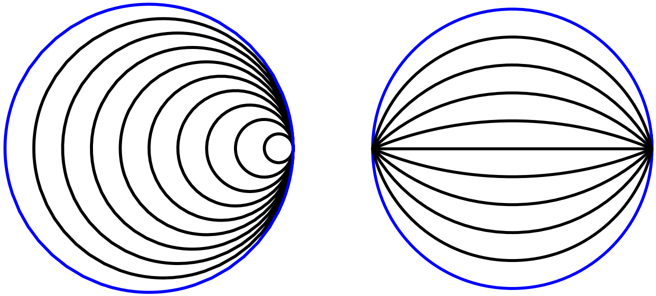
The two foliations on hyperbolic spaces





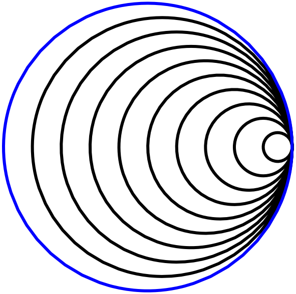
- horosphere foliation

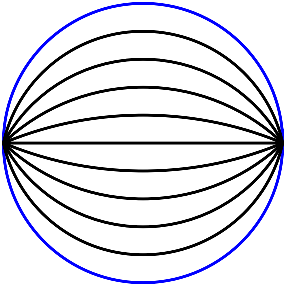
The two foliations on hyperbolic spaces



- horosphere foliation
- foliation with exactly one minimal leaf S

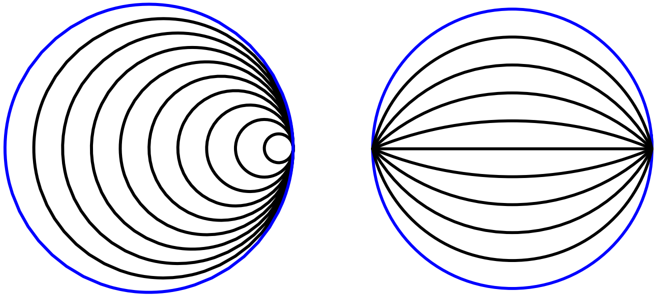
The two foliations on hyperbolic spaces





- horosphere foliation
- foliation with exactly one minimal leaf S
  - $\underline{M = \mathbb{R}H^n:}$ $S = \mathbb{R}H^{n-1}$ totally geodesic

The two foliations on hyperbolic spaces



- horosphere foliation
- foliation with exactly one minimal leaf S
  - $\underline{M = \mathbb{R}H^n}$: $S = \mathbb{R}H^{n-1}$ totally geodesic
  - $\underline{M = \mathbb{C}H^n}$: $S =$ ruled real hypersurface associated to a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$

Duality and triality

$\mathfrak{M}_F$ **depends only on the rank and on possible duality or triality principles on the symmetric space**

Duality and triality

$\mathfrak{M}_F$ **depends only on the rank and on possible duality or triality principles on the symmetric space**

Example: $\underline{r = 8,\ \mathrm{Aut}(DD) = I}$

Duality and triality

$\mathfrak{M}_F$ depends only on the rank and on possible duality or triality principles on the symmetric space

Example: $\underline{r = 8,\ \mathrm{Aut}(DD) = I}$

$$\boxed{\mathfrak{M}_F = \mathbb{R}P^7 \cup \{1, \ldots, 8\}}$$

for the following symmetric spaces:

Duality and triality

$\mathfrak{M}_F$ depends only on the rank and on possible duality or triality principles on the symmetric space

Example: $\underline{r = 8,\ \mathrm{Aut}(DD) = I}$

$$\boxed{\mathfrak{M}_F = \mathbb{R}P^7 \cup \{1, \ldots, 8\}}$$

for the following symmetric spaces:

$$SO_{17}^{\mathbb{C}}/SO_{17}\ , Sp_8^{\mathbb{R}}/U_8\ ,\ Sp_8^{\mathbb{C}}/Sp_8\ ,\ SO_{16}^{\mathbb{H}}/U_{16}\ ,\ SO_{17}^{\mathbb{H}}/U_{17}$$

$$E_8^8/SO_{16}\ ,\ E_8^{\mathbb{C}}/E_8$$

Duality and triality

$\mathfrak{M}_F$ depends only on the rank and on possible duality or triality principles on the symmetric space

Example: $\underline{r = 8,\ \mathrm{Aut}(DD) = I}$

$$\boxed{\mathfrak{M}_F = \mathbb{R}P^7 \cup \{1, \ldots, 8\}}$$

for the following symmetric spaces:

$$SO_{17}^{\mathbb{C}}/SO_{17}\ , Sp_8^{\mathbb{R}}/U_8\ ,\ Sp_8^{\mathbb{C}}/Sp_8\ ,\ SO_{16}^{\mathbb{H}}/U_{16}\ ,\ SO_{17}^{\mathbb{H}}/U_{17}$$

$$E_8^8/SO_{16}\ ,\ E_8^{\mathbb{C}}/E_8$$

and for the hyperbolic Grassmannians

$$G_8^*(\mathbb{R}^{n+16})\ (n \geq 1)\ ,\ G_8^*(\mathbb{C}^{n+16})\ (n \geq 0)\ ,\ G_8^*(\mathbb{H}^{n+16})\ (n \geq 0)$$

Geometry of the foliations

- $\ell \in \mathbb{R}P^{r-1} \leftrightsquigarrow$ horosphere foliation $\mathcal{F}_\ell$ on M

Geometry of the foliations

- $\ell \in \mathbb{R}P^{r-1} \leftrightsquigarrow$ horosphere foliation $\mathcal{F}_\ell$ on M
- all leaves of $\mathcal{F}_\ell$ are congruent to each other

Geometry of the foliations

- $\ell \in \mathbb{R}P^{r-1} \leftrightsquigarrow$ horosphere foliation $\mathcal{F}_\ell$ on M
- all leaves of $\mathcal{F}_\ell$ are congruent to each other
- if $r \geq 2$ then some foliations $\mathcal{F}_\ell$ are harmonic (i.e. all leaves are minimal) and hence induce a harmonic map $M \to \mathbb{R}$

Geometry of the foliations

- $\ell \in \mathbb{R}P^{r-1} \leftrightsquigarrow$ horosphere foliation $\mathcal{F}_\ell$ on M
- all leaves of $\mathcal{F}_\ell$ are congruent to each other
- if $r \geq 2$ then some foliations $\mathcal{F}_\ell$ are harmonic (i.e. all leaves are minimal) and hence induce a harmonic map $M \to \mathbb{R}$

- $i \in \{1, \ldots, r\} \leftrightsquigarrow \alpha_i \in \Lambda \leftrightsquigarrow$ foliation $\mathcal{F}_i$ on M

Geometry of the foliations

- $\ell \in \mathbb{R}P^{r-1} \leftrightsquigarrow$ horosphere foliation $\mathcal{F}_\ell$ on M
- all leaves of $\mathcal{F}_\ell$ are congruent to each other
- if $r \geq 2$ then some foliations $\mathcal{F}_\ell$ are harmonic (i.e. all leaves are minimal) and hence induce a harmonic map $M \to \mathbb{R}$

- $i \in \{1, \dots, r\} \leftrightsquigarrow \alpha_i \in \Lambda \leftrightsquigarrow$ foliation $\mathcal{F}_i$ on M
- $\mathcal{F}_i$ contains exactly one minimal leaf

Geometry of the foliations

- $\ell \in \mathbb{R}P^{r-1} \leftrightsquigarrow$ horosphere foliation $\mathcal{F}_\ell$ on M
- all leaves of $\mathcal{F}_\ell$ are congruent to each other
- if $r \geq 2$ then some foliations $\mathcal{F}_\ell$ are harmonic (i.e. all leaves are minimal) and hence induce a harmonic map $M \to \mathbb{R}$

- $i \in \{1, \ldots, r\} \leftrightsquigarrow \alpha_i \in \Lambda \leftrightsquigarrow$ foliation $\mathcal{F}_i$ on M
- $\mathcal{F}_i$ contains exactly one minimal leaf
- $|\alpha_i| = |\alpha_j| \Rightarrow \mathcal{F}_i, \mathcal{F}_j$ have the same principal curvatures with the same multiplicities

Geometry of the foliations

- $\ell \in \mathbb{R}P^{r-1} \rightsquigarrow$ horosphere foliation $\mathcal{F}_\ell$ on M
- all leaves of $\mathcal{F}_\ell$ are congruent to each other
- if $r \geq 2$ then some foliations $\mathcal{F}_\ell$ are harmonic (i.e. all leaves are minimal) and hence induce a harmonic map $M \to \mathbb{R}$

- $i \in \{1, \ldots, r\} \rightsquigarrow \alpha_i \in \Lambda \rightsquigarrow$ foliation $\mathcal{F}_i$ on M
- $\mathcal{F}_i$ contains exactly one minimal leaf
- $|\alpha_i| = |\alpha_j| \Rightarrow \mathcal{F}_i, \mathcal{F}_j$ have the same principal curvatures with the same multiplicities

Corollary: If $r \geq 3$, then there exist noncongruent homogeneous isoparametric systems on M with the same spectral data for the second fundamental form

Example

$$M = SL_4(\mathbb{R})/SO_4 = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \right\}$$

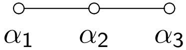
$x_{ij} \in \mathbb{R}$, $x_{11}x_{22}x_{33}x_{44} = 1$

Example

$$M = SL_4(\mathbb{R})/SO_4 = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \right\}$$

$x_{ij} \in \mathbb{R}$, $x_{11}x_{22}x_{33}x_{44} = 1$

$DD =$

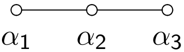


Example

$$M = SL_4(\mathbb{R})/SO_4 = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \right\}$$

$x_{ij} \in \mathbb{R}$, $x_{11}x_{22}x_{33}x_{44} = 1$

$$DD =$$



$$\mathcal{F}_1 \approx \left\{ \begin{pmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \right\} , \ \mathcal{F}_2 \approx \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \right\}$$