GEOMETRIC CONDITIONS LEADING TO PATTERSON-WALKER EXTENSION METRICS

Andrzej Derdzinski

The Ohio State University, Columbus, Ohio, USA

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If Σ is a manifold of any dimension, the cotangent bundle $M = T^*\Sigma$ carries a *canonical partial metric* $\langle\,,\rangle$, with

$$\langle \xi, w\rangle \;=\; \xi(d\pi_x w)$$

for $x \in M = T^*\Sigma$, vertical vectors $\xi \in \mathrm{Ker}\, d\pi_x = T_y^*\Sigma$, with $y = \pi(x)$, and any $w \in T_xM$. Here and below $\pi : T^*\Sigma \to \Sigma$ denotes the bundle projection.

The vertical distribution $\mathcal{V} = \mathrm{Ker}\, d\pi$ is $\langle\,,\rangle$-null, and $\mathrm{rank}\,\langle\,,\rangle = \dim\Sigma$. Thus, $\langle\,,\rangle$ constitutes a vector-bundle isomorphism $\mathcal{V}^* \to (TM)/\mathcal{V}$.

Any total-metric extension of $\langle\,,\rangle$ has the neutral signature. Every connection ∇ on Σ gives rise to such an extension g^∇, obtained by requiring all ∇-horizontal vectors to be g^∇-null.

In general, if $\mathcal{V} \subset TM$ is the tangent bundle of a foliation on a manifold M, the restriction of the normal bundle $(TM)/\mathcal{V}$ to each leaf carries the *Bott connection*, which is the canonical flat connection defined by requiring the images in $(TM)/\mathcal{V}$ of local $\mathcal{V}$-projectable vector fields in M to be parallel.

Note that u is $\mathcal{V}$-projectable if and only if $[u, v]$ is a (local) section of $\mathcal{V}$ whenever so is v.

Under our isomorphism $\langle\,,\rangle$ the Bott connection produces a flat connection on each leaf (fibre of $T^*\Sigma$). These flat connections have the additional property of being torsionfree, as they coincide with the standard flat connections on the vector spaces $T_y^*\Sigma$.

Let ∇ be a fixed torsionfree connection on a manifold Σ. The Patterson-Walker *Riemann extension metric* associated with an arbitrary symmetric 2-tensor τ on Σ is defined by

$$g \;=\; g^\nabla \,+\, 2\pi^*\tau\,.$$

Thus, $g = g^\nabla + 2\pi^*\tau$ is a pseudo-Riemannian metric of the neutral signature on $T^*\Sigma$. In local coordinates y^i, q_i for $T^*\Sigma$ arising from a coordinate system y^i for Σ in which ∇ has the components Γ^i_{jk},

$$g \;=\; 2\,dq_i \odot dy^i + 2(\tau_{jk} - q_i\Gamma^i_{jk})\,dy^j \odot dy^k.$$

Assuming that ∇ is torsionfree leads to no loss of generality.

Any 1-form ξ on Σ gives rise to a diffeomorphism $E_\xi : M \to M$, acting as the translation by ξ_y in each fibre $T_y^*\Sigma$, $y \in \Sigma$, and then

$$E_\xi^*(g^\nabla + 2\pi^*\tau) = g^\nabla + 2\pi^*(\tau + \mathcal{L}\xi),$$

where $\mathcal{L}$ is the *Killing operator* of ∇, sending 1-forms ξ on Σ to symmetric 2-tensors, with

$$2\mathcal{L}\xi = \nabla\xi + (\nabla\xi)^* \text{ that is, } (\mathcal{L}\xi)_{ij} = (\xi_{j,i} + \xi_{i,j})/2.$$

Thus, with fixed ∇ and variable τ, the Riemann extensions form a well-defined class of metrics on the total space of any affine bundle over Σ whose associated vector bundle is $T^*\Sigma$.

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For any Σ, ∇, τ as above, with $\dim \Sigma = m$, the *Riemann extension triple* $(M, g, \mathcal{V})$ formed by $M = T^*\Sigma$, $g = g^\nabla + 2\pi^*\tau$, and the vertical distribution $\mathcal{V} = \mathrm{Ker}\, d\pi$, has the following properties:

- (M, g) is a pseudo-Riemannian manifold of the neutral metric signature and $\dim M = 2m$,
- $\mathcal{V}$ is an n-dimensional null parallel distribution on (M, g),
- $\bar{R}(v, \cdot, v', \cdot) = 0$ for all sections v, v' of $\mathcal{V}$.

We use the symbols $\overline{\nabla}$ and $\bar{R}$ for the Levi-Civita connection of g and its curvature tensor.

As shown by Afifi (1954), the three properties just listed form an intrinsic local characterization of Riemann extensions triples.

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For $(M, g, \mathcal{V})$ arising from a triple (Σ, ∇, τ) as described above, both Σ and ∇ are local geometric invariants of g and $\mathcal{V}$ (while τ itself is not).

Specifically, Σ is the local leaf space of the foliation associated with $\mathcal{V}$.

As for ∇, it is determined by g and $\mathcal{V}$ as follows. Given π-projectable vector fields u, u' on M, the covariant derivative $\overline{\nabla}_u u'$ is π-projectable onto the vector field $\nabla_w w'$ on Σ, where w, w' are the π-images of u and u'.

Consequently, ∇ constitutes a transversal connection, in the sense of Molino (1968), for the foliation associated with $\mathcal{V}$.

MORE ON THE KILLING OPERATOR

For the Killing operator $\mathcal{L}$ of a torsionfree connection ∇ on a manifold Σ we always have

$$\dim \mathrm{Ker}\,\mathcal{L} \le 3,$$

since the Killing equation $\mathcal{L}\xi = 0$ implies that

$$\xi_{i,jk} = R_{ijk}{}^{s}\xi_s$$

via the Ricci identity.

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Let $\mathcal{S}_q$, $q \geq 0$, be the space of all symmetric q times covariant tensors of class C^∞ on Σ. There is the short exact sequence

$$0 \to \mathrm{Ker}\,\mathcal{L} \xrightarrow{\subseteq} \mathcal{S}_1 \xrightarrow{\mathcal{L}} \mathcal{S}_2.$$

Both $\mathrm{Ker}\,\mathcal{L}$ and the image $\mathcal{L}(\mathcal{S}_1)$ have geometric significance for Riemann extension metrics:

- elements of $\mathrm{Ker}\,\mathcal{L}$ naturally correspond to vertical Killing fields on a Riemann extension $(M, g) = (T^*\Sigma,\, g^\nabla + 2\pi^*\tau)$, and constitute fibre-preserving isometries of (M, g),

- the coset $[\tau] = \tau + \mathcal{L}(\mathcal{S}_1)$ in $\mathcal{S}_2$ is a geometric invariant of g (while τ itself is not). The pair $(\nabla, [\tau])$ determines g up to an isometry.

FIRST CASE: PARALLEL WEYL TENSOR OF RANK 1

This case involves:

- pseudo-Riemannian four-manifolds (M, g) such that the Weyl tensor of (M, g) is parallel, and, acting on 2-forms, has rank 1 at some/every point.

The local structure of such manifolds is known, also in higher dimensions (D. & Roter, 2007).

In dimension four, manifolds (M, g) with parallel, rank-one Weyl tensor are, locally, the same as the Riemann extension triples

$$(M, g, \mathcal{V}) = (T^*\Sigma, \, g^\nabla + 2\pi^*\tau, \, \mathrm{Ker}\, d\pi),$$

in which

- ∇ is a projectively flat torsionfree connection on the surface Σ with a fixed ∇-parallel area form α,
- τ satisfies a specific nonhomogeneous linear second-order partial differential equation $\mathcal{F}\tau = \pm\frac{1}{2}$ (see below).

Calling (M, g) "the same" as $(M, g, \mathcal{V})$ makes sense since, in this case, g determines $\mathcal{V}$. Specifically, $\mathcal{V}$ is the kernel (and image) of a 2-form that spans, locally, the image of the Weyl tensor.

The differential operator $\mathcal{F}$ appearing in the equation imposed on τ is given by

$$\mathcal{F}\tau \;=\; \mathrm{div}^{\nabla}\mathrm{div}^{\nabla}\Theta + (\mathrm{Ric}^{\nabla}, \Theta),$$

where Θ is the twice-contravariant symmetric tensor on Σ corresponding to τ under the bundle isomorphism $T\Sigma \to T^*\Sigma$ induced by α, and $(\,,\,)$ is the obvious pairing between covariant and contravariant 2-tensors. In coordinates,

$$\mathcal{F}\tau \;=\; \Theta^{ij}{}_{,ij} + R_{ij}\Theta^{ij}, \;\; \text{with} \;\; \tau_{ij} = \alpha_{ir}\alpha_{js}\Theta^{rs}.$$

Projective flatness of ∇ means that Ric^{∇} satisfies the Codazzi equation (or, equivalently, the 3-tensor $\nabla\mathrm{Ric}^{\nabla}$ is totally symmetric).

Projectively flat torsionfree surface connections with a parallel area form are completely understood: locally, such (Σ, ∇) consist of a surface Σ in $\mathbf{R}^3$ transverse to all lines through 0, and the *centroaffine connection* ∇ obtained in the standard way by declaring the radial directions normal to Σ.

radial = "normal"

Some of the four-manifolds having parallel, rank-one Weyl tensor are locally symmetric.

In terms of the centroaffine model, they correspond to open submanifolds of quadric surfaces centered at 0:

- planes not containing 0,
- elliptic and hyperbolic cylinders,
- ellipsoids,
- two-sheeted and one-sheeted hyperboloids.

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MORE ON THE FIRST CASE

There is a longer exact sequence (D. & Roter, 2007):

$$0 \to \mathrm{Ker}\,\mathcal{L} \overset{\subset}{\to} \mathcal{S}_1 \overset{\mathcal{L}}{\to} \mathcal{S}_2 \overset{\mathcal{F}}{\to} \mathcal{S}_0 \to 0$$

provided that Σ is noncompact and simply connected. Also, in this case, dim $\mathrm{Ker}\,\mathcal{L} = 3$ if Σ simply connected.
In the centroaffine model, elements of $\mathrm{Ker}\,\mathcal{L}$ are naturally identified with vectors in $\mathbf{R}^3$.

Fibre-preserving isometries act transitively on each fibre, and the isometry type of g does not depend on τ (while τ, locally, always exists). The local moduli of such (M,g) are in a one-to-one correspondence with those of pairs (∇,α).

THE FIRST CASE AND COMPACTNESS

It is not known whether a manifold (M, g) with rank-one, parallel Weyl tensor, of any dimension, can be compact without being locally symmetric.

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SECOND CASE: TYPE III SDNE WALKER METRICS

Here by an *SDNE manifold* we mean a self-dual neutral Einstein four-manifold, *type* III refers to its self-dual Weyl tensor acting on self-dual 2-forms, and the *Walker property* is the existence of a two-dimensional null parallel distribution compatible with the orientation.

A traceless endomorphism of a pseudo-Euclidean 3-space is said to be of *Petrov type* III if it

- is self-adjoint, and
- sends some ordered basis $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ to $(0, \mathbf{x}, \mathbf{y})$ (or, equivalently, its square is non-diagonalizable).

All type III SDNE Walker manifolds are Ricci-flat (Díaz-Ramos, García-Río and Vázquez-Lorenzo, 2006).

There also exist non-Walker type III SDNE manifolds. Their local structure, at points in general position, has been completely described (D., 2009). Some of them are Ricci-flat, some are not.

Type III SDNE manifolds are also known as the *type* III *Jordan-Osserman manifolds of dimension four*.

Type III SDNE metrics are known to be curvature homogeneous (Blažić, Bokan and Rakić, 2000). However, they need not, in general, be locally homogeneous.

The local structure of type III SDNE Walker manifolds (M, g) was determined by Díaz-Ramos, García-Río and Vázquez-Lorenzo (2006). Locally, they are the same as the Riemann extension triples

$$(M, g, \mathcal{V}) = (T^*\Sigma,\, g^\nabla + 2\pi^*\tau,\, \mathrm{Ker}\, d\pi),$$

in which the 2-tensor τ is arbitrary, while

- ∇ is a torsionfree connection on the surface Σ such that Ric^∇ is skew-symmetric and nonzero at each point.

Calling (M, g) "the same" as $(M, g, \mathcal{V})$ makes sense again, as g determines $\mathcal{V}$. Namely, the self-dual Weyl tensor acting on self-dual 2-forms has rank 2 at every point x, and we may declare $\mathcal{V}_x$ to be the nullspace of some/any, self-dual 2-form at x spanning its one-dimensional kernel.

SKEW-SYMMETRY OF THE RICCI TENSOR

The local structure of torsionfree surface connections with skew-symmetric Ricci tensor was described by Wong (1964):

WONG'S THEOREM (1964). *A torsionfree connection ∇ on a surface Σ has skew-symmetric Ricci tensor if and only if, on some neighborhood of any point of Σ, there exist coordinates in which the component functions of ∇ are $\Gamma_{11}^1 = -\partial_1\varphi$, $\Gamma_{22}^2 = \partial_2\varphi$ for a function φ, and $\Gamma_{jk}^i = 0$ unless $i = j = k$.*

The Ricci tensor of ∇ then is given by $R_{12} = -\partial_1\partial_2\varphi$.

Wong's paper provided three coordinate expressions which, locally, represent all torsionfree surface connections ∇ such that Ric^∇ is skew-symmetric and nonzero everywhere.

The above version is simplified: Ric^∇ is allowed to vanish, and the three coordinate forms are replaced with just one.

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Let ∇ be a torsionfree connection on a surface Σ such that Ric^∇ is skew-symmetric at every point.

Then $\dim \mathrm{Ker}\,\mathcal{L} \le 1$.

We call ∇ *generic* if $\mathrm{Ric}^\nabla \ne 0$ everywhere and the vector-bundle morphism

$$Q = 4 + \nabla w + \frac{3}{4}\phi \otimes w : T\Sigma \to T\Sigma$$

is injective at every point. Here ϕ and w are the 1-form and vector field defined by

$$\nabla \mathrm{Ric}^\nabla = \phi \otimes \mathrm{Ric}^\nabla, \qquad \phi = \mathrm{Ric}^\nabla(w, \cdot\,).$$

If ∇ is generic, then

$$\operatorname{Ker}\mathcal{L} = \{0\}$$

and there is a longer exact sequence:

$$0 \to \mathcal{S}_1 \xrightarrow{\mathcal{L}} \mathcal{S}_2 \xrightarrow{\mathcal{P}} \mathcal{S}_2 \xrightarrow{\mathcal{Z}} \mathcal{S}_1 .$$

Here $\mathcal{P}$ and $\mathcal{Z}$ are natural linear differential operators of orders 4 and 3.

In addition, $\mathcal{P} : \mathcal{S}_2 \to \mathcal{S}_2$ is the projection onto the second summand in a direct-sum decomposition

$$\mathcal{S}_2 = \mathcal{L}(\mathcal{S}_1) \oplus \operatorname{Ker}\mathcal{Z} .$$

Thus, each coset $[\tau] = \tau + \mathcal{L}(\mathcal{S}_1)$ has a unique representative lying in $\operatorname{Ker} \mathcal{Z}$ (which is equal to $\mathcal{P}\tau$).

We call a type III SDNE Walker manifolds (M, g) *generic* if so is the corresponding surface connection ∇.

The local moduli of such (M, g) are in a one-to-one correspondence with those of pairs (∇, σ), in which ∇ is a generic torsionfree surface connection with skew-symmetric Ricci tensor, and $\sigma \in \operatorname{Ker} \mathcal{Z}$.

In contrast with the first case, there is still no satisfactory description of local moduli for (M, g) (or (∇, σ)) as above.

$$[(\mathcal{B}\tau)(v)]\rho \;=\; [d^{\nabla}\tau](\,\cdot\,,\,\cdot\,,v),$$

$$2[\mathcal{D}\xi]\mathrm{Ric}^{\nabla} \;=\; \xi\wedge\phi \;-\; d\xi,$$

$$\mathcal{Z}\tau \;=\; 2d[\mathcal{D}(\mathcal{B}\tau)] + 4\mathcal{B}\tau - \tau(w,\,\cdot\,) + 3[\mathcal{D}(\mathcal{B}\tau)]\phi/2,$$

$$\mathcal{P}\tau \;=\; \tau \;-\; \mathcal{L}[(Q^{*})^{-1}\mathcal{Z}\tau].$$

THE SECOND CASE AND COMPACTNESS

All type III SDNE Walker manifolds are noncompact.

The reason is that every such manifold carries a natural vector field with nonzero constant divergence.

This vector field is vertical (a section of $\mathcal{V}$), and is "built" from the covariant derivative of the curvature tensor.

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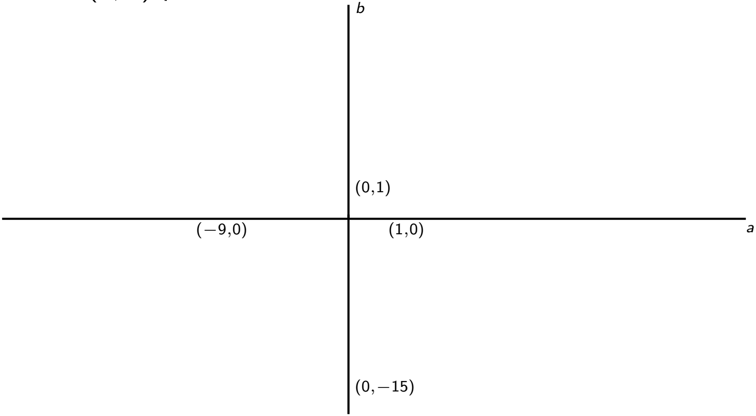
A MODULI CURVE

of nonflat locally homogeneous torsionfree surface connections with skew-symmetric Ricci tensor: the union of the coordinate axes in the (a, b)-plane $\mathbf{R}^2$.



A canonical coordinate form of such connections was first found by Kowalski, Opozda and Vlášek (2000). Here is a Lie-group description.

On a two-dimensional non-Abelian Lie group with a basis u, w of left-invariant vector fields such that $[u, w] = 2u$, the connection $\nabla = \nabla(a, b)$ is defined by

$$\begin{aligned} \nabla_u u &= (3 + a)u - aw, \\ \nabla_u w &= au + (3 - a)w, \\ \nabla_w u &= (a - 2)u + (3 - a)w, \\ \nabla_w w &= (a + b - 1)u + (2 - a)w. \end{aligned}$$

The degree of mobility equals 3 for $\nabla(1, 0)$, and 2 for all other $\nabla(a, b)$. All $\nabla(a, b)$ are generic, except for $\nabla(-9, 0)$, with dim $\mathrm{Ker}\,\mathcal{L} = 1$, and $\nabla(-15, 0)$, which has $\mathrm{Ker}\,\mathcal{L} = \{0\}$.

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A type III SDNE Walker manifold is never locally homogeneous, and its maximum possible degree of mobility is 3.

The argument uses what we know about vertical Killing fields and degrees of mobility of nonflat locally homogeneous torsionfree surface connections with skew-symmetric Ricci tensor.

Degree of mobility equal to 3 is realized only by $\nabla(1,0)$, with $\tau = 0$, and $\nabla(-9,0)$, with any left-invariant τ (which yields a one-parameter family of nonisometric examples).

This stands in contrast with the first case (rank-one, parallel Weyl tensor), where locally homogeneous examples are abundant.

OTHER DESCRIPTIONS OF $\nabla(1,0)$

In a two-dimensional real vector space Π with an area form Ω, we set, for a fixed real constant c, all vector fields u and all constant vector fields v on Π,

$$\nabla_u v \;=\; 2c[\Omega(w,u)v + \Omega(w,v)u] - c^2\Omega(w,u)\Omega(w,v)w.$$

Note that ∇ is invariant under the action of the unimodular group $\mathrm{SL}(\Pi)$. (Different choices of c lead to equivalent connections.)

For Π and Ω as above, $\mathrm{SL}(\Pi)$ acts by conjugation on the three-dimensional vector space V of all traceless endomorphisms of Π, and its action preserves the Lorentzian $(-++)$ inner product with the quadratic form $-\det$.

Thus, V is the Lie algebra of $\mathrm{SL}(\Pi)$, the action amounts to the adjoint representation, and the inner product is a multiple of the Killing form of $\mathrm{SL}(\Pi)$.

The action of $\mathrm{SL}(\Pi)$ is not effective: its kernel is the center $\mathbf{Z}_2$ of $\mathrm{SL}(\Pi)$, and $\mathrm{SL}(\Pi)/\mathbf{Z}_2$ acting on V coincides with the identity component $\mathrm{SO}^{\uparrow}(\mathrm{V})$ of the Lorentz group.

The $\mathrm{SL}(\Pi)$-equivariant quadratic mapping $\Phi : \Pi \to \mathrm{V}$ defined by $\Phi(y) = \Omega(y, \cdot\,) \otimes y$ sends $\Pi \smallsetminus \{0\}$ onto the *future null cone* Σ in V, which is a specific connected component Σ of the set of nonzero null vectors.

The future null cone Σ in V admits a one-parameter family of $\mathrm{SO}^{\uparrow}(\mathrm{V})$-invariant torsionfree connections and having nonzero, skew-symmetric Ricci tensor.

In fact, since $\Phi : \Pi \smallsetminus \{0\} \to \Sigma$ is a two-fold covering, we may choose the connections in question to be the Φ-images of the $\mathrm{SL}(\Pi)$-invariant connections ∇ on Π described earlier. The deck transformation $-\mathrm{Id} \in \mathrm{SL}(\Pi)$ leaves any such ∇ invariant.

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ANOTHER DESCRIPTION OF $\nabla(0,1)$

The connection $\nabla(0,1)$ is locally equivalent to a connection on a pseudoshphere (one-sheeted hyperboloid) S in a Lorentzian 3-space, invariant under a subgroup of the Lorentz group that leaves invariant a pair of parallel lines contained in S.

THE HOROCYCLE FOLIATION

Let Σ be, again, a future null cone in a Lorentzian 3-space V, and let Y be the sheet, adjacent to Σ, of the two-sheeted hyperboloid formed by all unit timelike vectors in V.

Thus, Y is a model of the hyperbolic plane.

The unit tangent bundle $T^1\mathrm{Y}$ may be identified with the submanifold of $\mathrm{Y} \times \mathrm{S}$ consisting of all orthogonal pairs $(p, q) \in \mathrm{Y} \times \mathrm{S}$. The formula $F(p, q) = p + q$ defines a mapping $F : T^1\mathrm{Y} \to \Sigma$ which is a fibration, as the connected Lorentz group $\mathrm{SO}^{\uparrow}(\mathrm{V})$ acts transitively on both $T^1\mathrm{Y}$ and Σ, while F is obviously $\mathrm{SO}^{\uparrow}(\mathrm{V})$-equivariant.

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The fibres of F are the leaves of the *horocycle foliation* on $T^1\mathrm{Y}$, called so because they are easily verified to be the natural lifts to $T^1\mathrm{Y}$ of oriented horocycles in Y.

The horocycle foliation descends from $T^1\mathrm{Y}$ to the unit tangent bundle $T^1\Sigma$ of any closed orientable surface Σ of genus greater than 1 endowed with a hyperbolic metric.

This is due to its invariance under the action on $T^1\mathrm{Y}$ of the group $\mathrm{SO}^\uparrow(\mathrm{V})$ of all orientation-preserving isometries of the hyperbolic plane Y.

The invariance follows in turn from the $\mathrm{SO}^\uparrow(\mathrm{V})$-equivariance of F, mentioned above.

The horocycle foliation on the unit tangent bundle $T^1\Sigma$ of any closed orientable surface Σ of genus greater than 1, for any hyperbolic metric on Σ, admits a transversal torsionfree connection with everywhere-nonzero, skew-symmetric Ricci tensor.

An obvious Cartesian-product construction shows that such transversal connections exist on suitable compact manifolds with codimension-two foliations in all dimensions $n \geq 3$.

However, they do not exist in dimension 2.