Conformal actions of nilpotent groups on pseudo-Riemannian manifolds

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Outline

Setting

- (M, g) is a compact *semi-Riemannian manifold*: g is a symmetric, nondegenerate, bilinear form on TM
- $\dim M \geq 3$
- The *conformal class* $[g]$ of g comprises all metrics of the form $e^\sigma g$, for
$$\sigma : M \to \mathbf{R}$$
- $\mathrm{Conf}\, M$ consists of all $f \in \mathrm{Diff}\, M$ preserving the conformal class of g:
$$f^* g \in [g]$$

Lichnerowicz conjecture

M a compact Riemannian manifold

Fact: Isom M is compact.

Theorem (Lelong-Ferrand)

If Conf M is not compact, then M is conformally equivalent to the round sphere S^n.

Lichnerowicz conjecture, continued

For (M, g) semi-Riemannian, say $H < \text{Conf } M$ is *essential* if it does not preserve any metric $g' \in [g]$.

Conjecture (Semi-Riemannian Lichnerowicz)

If Conf M is essential, then M is locally conformally equivalent to flat $\mathbf{R}^{p,q}$.

Example: Einstein spaces

- Q standard quadratic form on $\mathbf{R}^{p+1,q+1}$:

$$Q(\mathbf{x}) = -\sum_{i=0}^{p} x_i^2 + \sum_{i=p+1}^{p+q+2} x_i^2$$

  The group of linear transformations of $\mathbf{R}^{p+1,q+1}$ preserving Q is $O(p+1, q+1)$
- $\mathcal{N}$ nullcone : $\{\mathbf{x} \in \mathbf{R}^{p+1,q+1} \mid Q(\mathbf{x}) = 0\}$
- $\mathrm{Ein}^{p,q} = \mathbf{P}(\mathcal{N}) \sim (S^p \times S^q)/\mathbf{Z}_2$
- $\mathrm{Conf}\ \mathrm{Ein}^{p,q} = PO(p+1, q+1)$
- Stabilizer is parabolic subgroup $P \cong CO(p,q) \ltimes \mathbf{R}^{p+q}$

Simple groups of conformal flows

Fact: $O(p+1, q+1)$ is simple, of rank $p+1$ when $p \leq q$.

Let $G < \text{Conf } M$ simple and connected.

Theorem (Zimmer)

$rk\ G \leq p+1$. If $G < Isom(M, g')$, then $rk\ G \leq p$.

Corollary

If $rk\ G = p+1$, then G is essential.

Theorem (Bader & Nevo; Frances & Zeghib)

If $rk\ G = p+1$, then M is conformally equivalent to $Ein^{p,q}$, up to finite covers when $p \geq 2$, up to finite and cyclic covers when $p = 1$.

Nilpotent groups

- For a group G, define

$$G_1 = [G, G] \qquad G_{k+1} = [G, G_k]$$

- G is *nilpotent* if $G_d = 1$ for some d; the *degree of nilpotence* $d(G)$ is the minimal such d
- **Fact:** If $G < O(p+1, q+1)$ is connected and nilpotent, then $d(G) \leq 2p + 1$.

Statement

$G < \mathrm{Conf}\ M$ connected and nilpotent.

Theorem (FM)

$d(G) \leq 2p+1$. If $d(G) = 2p+1$, then M is conformally equivalent to $\mathrm{Ein}^{p,q}$, up to finite covers when $p \geq 2$, up to finite and cyclic covers when $p = 1$.

Note: If $G < \mathrm{Isom}(M, g')$, then $d(G) \leq 2p$ (BFM), so if $G < \mathrm{Conf}\ M$ with $d(G) = 2p+1$, then G is essential.

Cartan geometries

A Cartan geometry infinitesimally models a manifold M on a homogeneous space G/P.

Definition

A Cartan geometry modeled on G/P is (M, B, ω) where

- $\pi : B \to M$ *is a principal P-bundle*
- ω *is a $\mathfrak{g}$-valued 1-form on B (satisfying 3 axioms)*

A type-(p, q) structure $(M, [g])$ corresponds to a canonical Cartan geometry modeled on $\mathrm{Ein}^{p,q}$ such that

$$\mathrm{Conf}\, M \cong \mathrm{Aut}(M, B, \omega)$$

Given $b \in \pi^{-1}(x)$, have *isotropy homomorphism*

$$\iota_b : \mathrm{Stab}(x) \hookrightarrow P$$

Exponential Map

$$\begin{aligned} \exp &: B \times \mathfrak{g} \to B \\ \exp(b, tX) &= \gamma_X(t) \quad \text{where} \\ \gamma_X(0) = b \ &\text{and} \ \omega(\gamma_X'(t)) \equiv X \end{aligned}$$

If $f \in \mathrm{Aut}(M, B, \omega)$, then

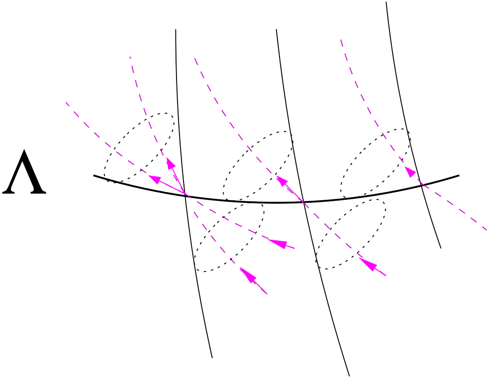
$$f(\exp(b, X)) = \exp(f(b), X)$$

If $f \in \mathrm{Stab}(x)$ with isotropy $\iota_b(f) = g \in P$, can relate dynamics of f near x with g near $[1] \in G/P$, using curves of the form

$$\pi \circ \exp(b, tX)$$

Null translations

On $\mathrm{Ein}^{1,n}$, exists conformal flow φ^t fixing a curve Λ, acting locally like



Have analogous flows on $\mathrm{Ein}^{p,q}$, $p > 1$.

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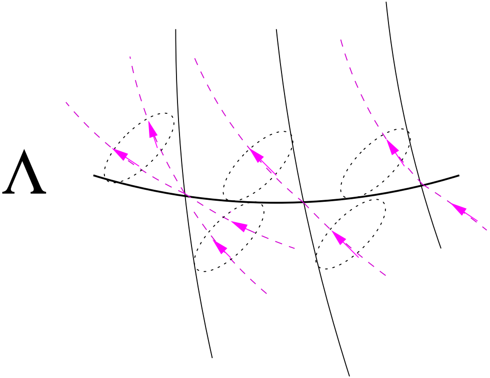

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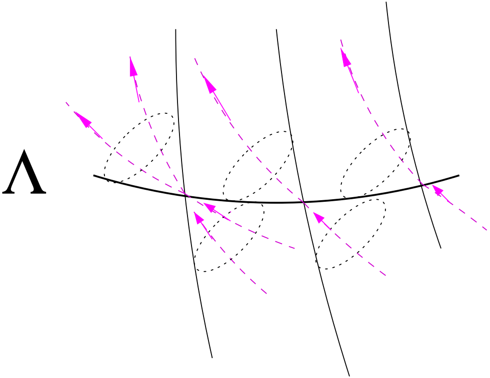


Have analogous flows on $\mathrm{Ein}^{p,q}$, $p > 1$.

Null translations, continued

- The fixed curve Λ is a *null geodesic* of $\mathrm{Ein}^{p,q}$, of the form $\pi_G \circ \exp(1, tX)$.
- The flow preserves *null cones* of each $x \in \Lambda$, consisting of all null geodesics emanating from x.
- In a cone, each flow trajectory is a null geodesic.

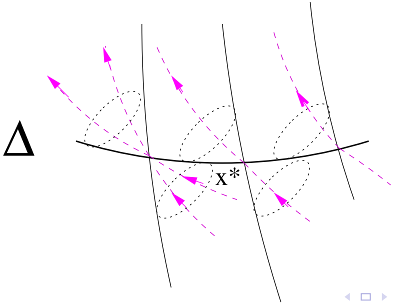
Proof Outline

1. $d(H) \leq 2p+1$, and null translation in isotropy if $d(H) = 2p+1$, using embedding theorem of BFM
2. Weyl curvature vanishes along null cones near fixed point
3. Global flatness
4. Developing map $\widetilde{M} \to \widetilde{\mathrm{Ein}}^{p,q}$ is a diffeomorphism, using Frances' boundary rigidity for embeddings of Cartan geometries

Steps 1, 3, and 4 use $d(H) = 2p+1$; step 2 uses only null translation in isotropy.

Flatness on an open set

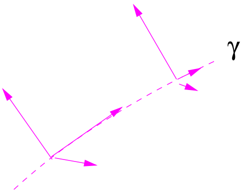
- Using embedding theorem of BFM for the associated Cartan geometry, show there exists $x_* \in M$ and flow $\varphi^t \in \mathrm{Stab}(x_*)$ with isotropy $\iota_b(\varphi^t)$ a flow by null translations.
- Via Cartan connection ω, can show φ^t has these dynamics around x_*.



Contracted framing

Let $\gamma(t) = \varphi^t(x_0)$ with $\lim_{t\to\infty} \varphi^t(x_0) = x_*$. Exists framing along γ in which

$$\varphi^t_* \sim \begin{pmatrix} 1 & & & & \\ & \lambda^{-t} & & & \\ & & \ddots & & \\ & & & \lambda^{-t} & \\ & & & & \lambda^{-2t} \end{pmatrix}$$



Weyl curvature

Weyl curvature

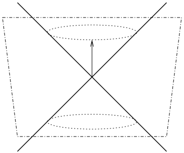
- tensor of type $(3, 1)$
- conformally invariant
- obstruction to conformal flatness

$$\begin{aligned} W_{\gamma(t)}(\varphi^t_* X_i, \varphi^t_* X_j, \varphi^t_* X_k) &= \varphi^t_* W_{x_0}(X_i, X_j, X_k) \\ \lambda^{\sigma(i,j,k)t} W_{\gamma(t)}(X_i, X_j, X_k) &= \varphi^t_* W_{x_0}(X_i, X_j, X_k) \end{aligned}$$

where $\sigma(i,j,k) = \rho(i) + \rho(j) + \rho(k)$, each $\rho(i) \in \{0, -1, -2\}$.

Vanishing of W

$$\lambda^{\sigma(i,j,k)t} W_{\gamma(t)}(X_i, X_j, X_k) = \varphi_*^t W_{x_0}(X_i, X_j, X_k)$$
$$\text{Im } W_{x_0} \subset X_n^{\perp}$$

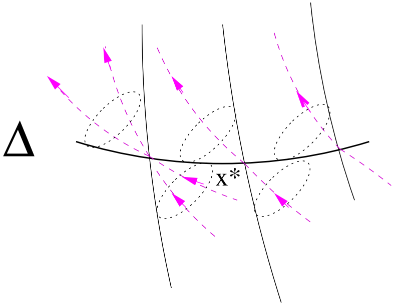


$$W_{x_*} = 0$$

$$W_{\gamma(t)} = 0$$

along all γ in the cone of x_*.

Vanishing of W, continued



The isotropy of φ^t at any point of Δ is same null translation. So $W = 0$ on all null cones from $\Delta \Rightarrow W = 0$ on an open set.

Further results

M is a type-(p,q) semi-Riemannian manifold.

- Boundary rigidity (Frances): Let $\Omega \subseteq \mathrm{Ein}^{p,q}$ with Hausdorff dimension $\partial\Omega < p+q-1$. Let $\Gamma\backslash\Omega \hookrightarrow M$ a conformal embedding. Then $M \cong \Gamma\backslash\Omega'$ with $\Omega \subseteq \Omega' \subseteq \mathrm{Ein}^{p,q}$.
- Local conformal flows (Frances & M): Let X be a local conformal vector field vanishing at $x_0 \in M$. If $\{(\varphi_X^t)_{*x_0} : t \in \mathbf{R}\}$ is precompact, then either φ_X^t is proper and complete, or M contains $\emptyset \neq U$ open and conformally flat with $x_0 \in \overline{U}$.