# Berger-type theorems

Carlos Olmos

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1. Introduction.
2. Connections, parallel transport and holonomy.
3. Riemannian holonomy.
4. Submanifold geometry and holonomy.
5. Holonomy systems and skew-torsion holonomy systems.
6. Applications to naturally reductive spaces.

In this talk we would like to draw the attention into some results that have common features and that we call Berger-type theorems.

Some of these results refer to submanifold geometry, some to Riemannian geometry and some of them refer to algebraic objects. They have very strong or interesting applications.

We do not pretend to define what a Berger-type theorem is. But, informally, it means:

If an orthogonal group is "generic"then our object is "symmetric".

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# Connections in vector bundles.

Let $E \xrightarrow{\pi} M$ be a vector bundle with a metric $\langle\ ,\ \rangle$ on the fibers $E_q = \pi^{-1}(\{q\})$, $q \in M$, and a metric connection $\nabla$. We will be particularly interested in the following two cases:

$$E = TM \quad (\nabla \text{ the Levi-Civita connection}).$$

the tangent bundle of a (pseudo-)Riemannian manifold, or

$$E = \nu M \quad (\nabla = \nabla^{\perp} \text{ the normal connection}).$$

the normal bundle of a submanifold of a space form (real or complex).

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Associated to any curve $c$ on $M$, from $p$ to $q$, one has the parallel transport $\tau_c : E_p \longrightarrow E_q$, which is a linear isometry.

The holonomy group $\Phi$ of $\nabla$ at $p \in M$ is a Lie subgroup of $SO(E_p)$ given by

$$\Phi_p = \{\tau_c : c \text{ is a loop by } p\}$$

Its connected component $\Phi_p^*$ is the so-called restricted holonomy group which coincides with the holonomy group of the universal cover.

Holonomy groups at two different points are conjugated by the parallel transport along any curve that connect them.

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# Riemannian holonomy.

If $M$ is a Riemannian manifold and $E = TM$, then the de Rham decomposition theorem implies that the (restricted) holonomy group acts irreducibly, unless $M$ is a Riemannian product (locally, or globally if $M$ is simply connected and complete).

Moreover, one has the well-known result of Marcel Berger (1955).

**Berger Holonomy Theorem.** *Assume that the (restricted) holonomy group of an irreducible Riemannian manifold is not transitive on the sphere. Then $M$ is locally symmetric.*

The above theorem follows from the classification of Berger of the possible holonomies of locally non-symmetric spaces. Namely,

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The above theorem follows from the classification of Berger of the possible holonomies of locally non-symmetric spaces. Namely,

- *SO*($n$), *generic Riemannian manifold* of dimension $n$.
- *U*($n$), *generic Khäler manifold* of real dimension $2n$.
- *Sp*($n$) × *Sp*(1), *generic quaternionic Khäler manifold*, dim $= 4n$ (always Einstein).
- *Spin*(9), always symmetric, *the Cayley plane or its dual*, dim $= 16$ (Dmitri Alekseevsky).
- *SU*($n$), *Calabi-Yau manifolds*, dim $= 2n$ (Khäler)
- *Sp*($n$), generic hyperkhäler manifold, dim $= 4n$ (Khäler and Ricci flat) .
- $G_2$, the so-called $G_2$-*manifolds*, dim $= 7$ (Ricci flat).
- *Spin*(7), the so-called *Spin*(7)-*manifolds*, dim $= 8$ (Ricci flat).

The last four groups in the list are the so-called exceptional holonomies, since there are no symmetric spaces having them as holonomy.

- $SO(n)$, *generic Riemannian manifold* of dimension $n$.
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The idea of classifying geometries in terms of their parallel tensors or spinors (which can be read off from the holonomy) goes back to Lichnerowicz.

Marcel Berger succeed in giving such a classification, which turned out to be very important since associated to any holonomy there is a well-known and famous geometry!.

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In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, $G$ is a Lie subgroup of $SO(\mathbb{V})$ and $R$ is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if $G$ acts irreducibly on $\mathbb{V}$.
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In 1962 James Simons succeed in giving a classification free proof of Berger theorem by defining the so-called holonomy systems. . This is triple

$$[\mathbb{V}, R, G]$$

where $\mathbb{V}$ is a Euclidean vector space, $G$ is a Lie subgroup of $SO(\mathbb{V})$ and $R$ is an algebraic Riemannian curvature tensor on $\mathbb{V}$ with values $R_{x,y} \in \mathcal{G}$.
Such a triple is said

irreducible, if $G$ acts irreducibly on $\mathbb{V}$.
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# Euclidean submanifold geometry

A similar rôle as symmetric spaces play in Riemannian geometry is played by all the orbits of $s$-representations i.e., the isotropy representation of semisimple symmetric spaces.

For example the isotropy representation of the symmetric space $SL(n)/SO(n)$ can be regarded as $SO(n)$ acting by conjugation on the space of traceless symmetric matrices.

The isotropy representation of the Grassmannian $SO(n+k)/SO(n) \times SO(k)$ can be regarded as the action of $SO(n) \times SO(k)$ on $\mathbb{R}^{n \times k}$ given by

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A symmetric space is characterized by the property that the parallel transport $\tau_c$ along any curve $c$, starting at $p$ is achieved by the differential of an isometry, i.e.,

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On the other hand, an orbit $M = K.v \subset \mathbb{R}^N$ of an $s$-representation is characterized by a similar property, with respect to the normal connection. Namely, the normal parallel transport $\tau_c^\perp$, along any $c$ in $M$, starting at $p$, can be achieved by the differential of some extrinsic isometry of $M$. That is, there exists an isometry of $\mathbb{R}^N$ such that $g(M) = M$ and

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**Informal Remark.** The normal connection gives weaker information, in submanifold geometry, than the Levi-Civita connection in Riemannian Geometry.

**Normal Holonomy Theorem**(O.) *The normal holonomy group of an Euclidean submanifold, acts on the normal space, up to its fixed point set, as the isotropy representation of semisimple symmetric space.*

The holonomy of a symmetric space, without Euclidean factor, coincides with the isotropy (represented on the tangent space). So, the normal holonomy theorem can be phrased as follows: *the normal holonomy representation of an Euclidean submanifold coincides with the holonomy representation of a symmetric space*. This means that the normal holonomy representation coincides with a non-exceptional riemannian holonomy.

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The above theorem is not a *Berger-type theorem*, since it gives information only about the representation of the normal holonomy on the normal space, but not about the space (in this case the submanifold).

As we informally pointed out, the normal holonomy gives weaker information than the Riemannian holonomy. So, interesting applications of the normal holonomy, can only be given within a restrictive class of submanifolds, as, for instance, the following:

- *Homogeneous submanifolds.*
- *Submanifolds with constant principal curvatures.*
- *Complex submanifolds.*

In order to illustrate this, we ennounce two Berger-type theorems, for the last two classes of submanifolds.

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The first one is a reformulation of a remarkable result of G. Thorbergsson about the homogeneity of isoparametric submanifolds of higher codimension (which can also be proven by using normal holonomy methods).

**Theorem** (Thorbergsson). Let $M$ be a submanifold of the sphere with constant principal curvatures. Assume that the normal holonomy group of $M$ acts irreducibly and non-transitively. Then $M$ is an orbit of an $s$-respresentation.

**Theorem** (Console, Di Scala, O.) Let $M$ be a complete and full complex submanifold of $\mathbb{C}P^n$. If the normal holonomy of $M$ is not transitive, then $M$ is the (unique) complex orbit, in the projectivezed space, of an irreducible Hermitian $s$-representation (or equivalently, $M$ is an extrinsic symmetric submanifold of $\mathbb{C}P^n$).

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A central result, related to normal holonomy, is the so-called Rank Rigidity theorem for submanifolds.

Decompose the normal bundle of $M$ as

$$\nu M = \nu_0 M \oplus \nu_0^{\perp} M$$

where $\nu_0 M$ is the maximal parallel and flat subbundle of $\nu M$.

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*The precise structure of the holonomy group of Riemannian submanifolds of Lorentz space was recently given by Kordian Lärs, by associating screen representations to normal holonomy groups (and classifying such representations with the Borel-Lichnerowicz property).* Such a precise description should have important applications in Lorentzian submanifold geometry.

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The following corollary is a key fact for relating normal holonomy with tangent holonomy. More precisely, for giving a geometric proof of Berger holonomy theorem (O., Ann. of Mah. 2005).

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This is a triple,

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In this case the triple $[\mathbb{V}, \theta, G]$ is called a skew-torsion holonomy system.

Such a $\mathcal{G}$-valued 1-form arises, usually, as the difference tensor between two different metric connections with the same geodesics. The so-called connections with skew-torsion (when one of the connections is the Levi-Civita connection). We will refer later to this.

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**Decomposition of skew-torsion holonomy systems:** Let $G'$ be the subgroup of $G$ with Lie algebra

$$\mathcal{G}' = \{g(\theta)_x : g \in G, \ x \in \mathbb{V}\}$$

One has, as it is standard to prove,

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k \qquad \text{(orthogonally)}$$

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where $G'_i$ acts irreducibly on $\mathbb{V}_i$ and trivially on $\mathbb{V}_j$, $i \neq j$ (Agricola-Friedrich).

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*Unlike the case of holonomy systems, no transitive groups can occur, except the full orthogonal group. In fact, the transitive cases were essentially disregarded, case by case, by Agricola-Friedrich (Math. Ann., 2004). Let us see our last Berger-type theorem.*

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**Skew-torsion Holonomy Theorem** (Nagy; O.-Reggiani). *Let* $[\mathbb{V}, \Theta, G]$, $\Theta \neq 0$, *be an irreducible skew-torsion holonomy system with* $G \neq \mathrm{SO}(\mathbb{V})$. *Then* $[\mathbb{V}, \Theta, G]$ *is symmetric and non-transitive. Moreover,*

1. $(\mathbb{V}, [\ ,\ ])$ *is an orthogonal simple Lie algebra, of rank at least 2, with respect to the bracket* $[x, y] = \Theta_x y$;
2. $G = \mathrm{Ad}(H)$, *where* $H$ *is the connected Lie group associated to the Lie algebra* $(\mathbb{V}, [\ ,\ ])$;
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This theorem was independently proved by Paul-A. Nagy by using the classification of the so-called Berger algebras, though in this final form a little earlier.

Our approach is geometric, based on submanifold geometry, and does not use any classification result. Our motivation for finding such a result came from homogenous spaces. In particular for explaining the inextendibility of the presentation group of an isotropy irreducible space which is not the sphere.

In the final part of this talk we will show these applications. First we will do some remarks for motivating why the only transitive group is the full orthogonal group.

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From a skew-torsion holonomy system $[\mathbb{V}, \theta, G]$ one can construct a holonomy system $[\mathbb{V}, R, G]$, with $sc(R) \neq 0$

Namely, by defining

$$R_{x,y} = [\theta_x, \theta_y] - \frac{2}{3}(\theta_x.\theta)_y$$

Observe that

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In some sense, skew-torsion holonomy systems are to holonomy systems what compact Lie groups are to compact symmetric spaces.

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# Nautrally reductive spaces.

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a $G$-invariant metric $\langle \ , \ \rangle$. The space $M$ is said to be naturally reductive if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m}$$

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such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$$

In other words, the geodesics of $M$ coincides with the $\nabla^c$-geodesics (i.e., the geodesics with respect the canonical connection $\nabla^c$, associated to the reductive decomposition).

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$$\nabla_v \tilde{w} = \frac{1}{2}[\tilde{v}, \tilde{w}]_p$$

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where, for $u \in T_pM$, $\tilde{u}$ is the Killing field on $M$ induced by the unique $X \in \mathfrak{m}$ such that $X.p = u$.

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Then

$$\theta_v w := D_v w - D^g_v w = -\frac{1}{2}\nabla_v Z$$

where $Z = \tilde{w} - \tilde{w}^g$.

Observe that $Z_p = 0$. Then $\nabla Z$ lies in the (full) isotropy algebra. Thus $\theta$ takes values in the isotropy algebra.

Combining this last observation with the skew- torsion holonomy theorem and other geometric arguments one has the following.

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**Theorem** (O.-Reggiani). *Let $M$ be a compact naturally reductive Riemannian manifold, which is locally irreducible. Assume furthermore that $M$ is neither (globally) isometric to a sphere, nor to a real projective space, nor to a compact simple Lie group with a bi-invariant metric. Then the canonical connection is unique.*

**Corollary** (O.-Reggiani). *Let $M = G/H$ be a naturally reductive Riemannian manifold and let $\nabla^c$ be the associated canonical connection. Assume that $M$ is locally irreducible and that $M \neq S^n$, $M \neq \mathbb{R}P^n$. Then*

1. $\mathrm{Iso}_0(M) = \mathrm{Aff}_0(M, \nabla^c)$.
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The above theorem has also the following corollary that explains, in a geometric way, that the presentation of an isotropy irreducible space gives the connected component of the full isometry group. This question was posed by J. Wolf, for the strongly isotropy irreducible spaces, and by M. Wang and W. Ziller in general.

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For a compact normal homogeneous space (i.e. $\mathfrak{m} = \mathfrak{h}^\perp$), with $G$ simple, this result was obtained by classification by Onishchik. For homogeneous manifolds of positive curvature the full isometry group was determined by Shankar.

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By usisng the previos theorem, Silvio Reggiani computed explicitly (the connected component of) the full isometry group of a normal homogeneous space. Namely, if $M = G/H$ is such a space and $S$ is the (connected component of the) fixed set of the isotropy $H$, regarded as a Lie group, then

$$\mathrm{Iso}(M)_0 = G_1 \times S$$

where $G_1$ is the semisimple part of $G$.

For the general naturally reductive case one has to replace $S$ by an appropriate subgroup.

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