# Hidden symmetries and the wave equation on Kerr

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joint work with Pieter Blue

# Outline

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1. Background
2. Hidden symmetries
3. Cut and paste energy
4. Generalized Morawetz
5. Decay estimate
6. Concluding remarks

# Isolated systems in GR

- The Einstein equations of General Relativity

  $$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

  relate the Lorentzian geometry of spacetime $(\mathcal{M}, g_{\alpha\beta})$ to matter fields with energy-momentum tensor $T_{\alpha\beta}$.
  - Isolated systems in GR give an idealized picture of eg. stars, galaxies, clusters.
  - Steady states are described by geometries which admit an (asymptotically) timelike Killing field, these are *stationary spacetimes*
  - If there is a *trapped region* which is invisible to observers at infinity, the spacetime contains a *black hole*
- The Schwarzschild and Kerr spacetimes are examples of black hole spacetimes. The Schwarzschild spacetime is in addition spherically symmetric and static, while Kerr is axially symmetric.

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# Black hole stability

- An isolated system in GR is (expected to be) asymptotically stationary
- The Kerr solution is (expected to be) the unique stationary, asymptotically flat, vacuum spacetime.
- Kerr describes a rotating black hole
  - parameters $a, M$, $0 \leq a \leq M$. $a \leftrightarrow$ angular momentum per unit mass, setting $a = 0$ gives Schwarzschild
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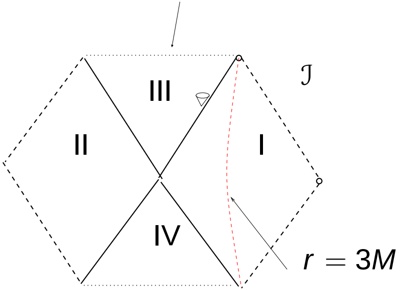
# Schwarzschild

$$g_{\alpha\beta}dx^\alpha dx^\beta = -fdt^2 + f^{-1}dr^2 + r^2 h_{S^2}, \quad f = 1 - \frac{2M}{r}$$

Singularity: $r = 0$



Maximally extended Schwarzschild

# Schwarzschild

- For Schwarzschild, decay estimates are known for scalar waves (Blue & Sterbenz, 2006; Dafermos & Rodnianski, 2005b) and Maxwell (Blue, 2008). See also eg. (Finster, Kamran, Smoller, & Yau, 2006) for an independent approach to wave equations on Schwarzschild and Kerr.
- Vector fields method: make use of momenta $P^\alpha(u, X)$ and deformation terms $\mathcal{T}_{\alpha\beta} D^\alpha X^\beta$ for cleverly chosen vector fields $X$ (including Morawetz trapping $\mathbf{A} = \mathcal{F}\partial_r$ and conformal $\mathbf{K} = u_+^2 \partial_+ + u_-^2 \partial_-$) — need lots of symmetries
- The proofs makes use of properties of the photon sphere in Schwarzschild (codimension one), and the spherical symmetry of the spacetime.
- Get decay $|u| \lesssim 1/t$ in stationary regions

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# Kerr

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\begin{aligned}
\Delta &= r^2 - 2Mr + a^2, \\
\Sigma &= r^2 + a^2 \cos^2\theta, \\
\Pi &= (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta
\end{aligned}
$$

and let $r_\pm$ denote the roots of $\Delta$,

$$r_\pm = M^2 \pm \sqrt{M^2 - a^2}$$

On the exterior region $r \geq r_+$, the Kerr metric can be written (Boyer & Lindquist, 1967)

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\begin{aligned}
g_{\mu\nu} dx^\mu dx^\nu = &- \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra \sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\
&+ \frac{\Pi \sin^2\theta}{\Sigma} d\phi^2
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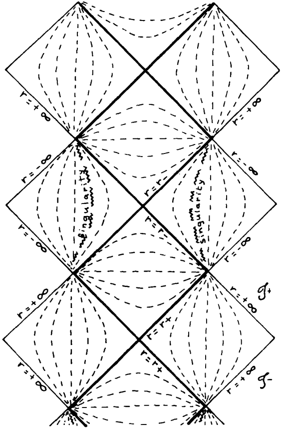

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# Kerr

- Kerr has a complicated photon sphere, and is only axi-symmetric
- Kerr has an ergoregion outside the horizon, where $\partial_t$ is spacelike $\Rightarrow$ there is no positive definite conserved energy.
- Kerr has a hidden symmetry related to the Carter constant – the geodesic equation on Kerr is separable
- Recent work using Fourier techniques provides decay estimates (Tataru & Tohaneanu, 2008; Dafermos & Rodnianski, 2008)
- Results presented here make use of “physical space” techniques

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# Main result

We can summarize the main result being presented here as follows.

**Theorem (Andersson & Blue, 2009)**

*Let u solve $\Box_g u = 0$ on the exterior region $\{r \geq r_+\}$ of the Kerr black hole spacetime, with inital data $[u^0]$ at $t_0$. There is an $a_0 > 0$ such that for $a \in [0, a_0]$*

1. *there is a norm $||[u^0]||_E$ on initial data and a constant $c_E$ such that for all t, the energy bound*
$$||u(t)||_{H^1} + ||\dot{u}(t)||_{L^2} \leq c_E ||[u^0]||_E$$
*holds on $\{r \geq r_+\}$,*
2. *there is a norm $||[u^0]||_B$ on initial data and constants $c_B, c_K$ such that for all t, the decay estimate*
$$||u(t)||_{L^\infty;loc} \leq c_B (1 + |t|)^{-1+c_K a} ||[u^0]||_B \,.$$
*holds on $\{r \geq r_+\}$.*

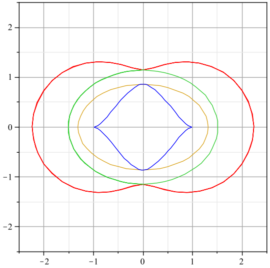
# Main result

### Remark

1. *This result corresponds to the decay results for waves in Schwarzschild, with a loss linear in a.*
2. *More detailed analysis shows that we have decay at the horizon $r = r_+$ and at $\mathcal{I}$ as in Schwarzschild, with a loss as above.*
3. *For the spherically symmetric case, one has $1/t^3$ decay, cf. (Dafermos & Rodnianski, 2005a) — expect analogous result to hold on Kerr.*

# Main difficulties

Ergoregion $\Rightarrow$ no globally timelike Killing field $\Rightarrow$ no positive definite conserved energy



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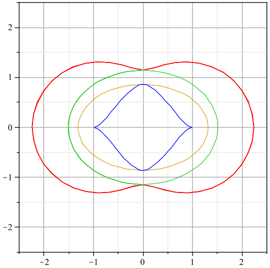

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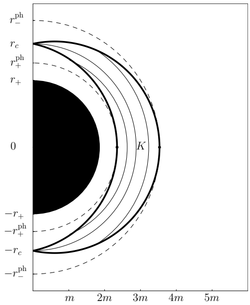
Photon sphere $\Rightarrow$ trapping for null geodesics and waves



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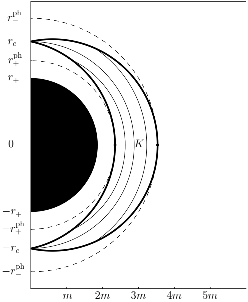

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# The Kerr wave operator

$$\Box = \Sigma \Box_g = \partial_r \Delta \partial_r + \frac{1}{\Delta} \mathcal{R}$$

where $\Box_g = \nabla^\alpha \nabla_\alpha$, and $\mathcal{R} = \mathcal{R}(r, \mathcal{E}, \mathcal{L}_z, \mathcal{Q}) = \mathcal{R}(r, \partial_t, \partial_\phi, Q)$ is

$$\mathcal{R} = -(r^2 + a^2)^2 \partial_t^2 - 4aMr \partial_t \partial_\phi + \Delta Q + (\Delta - a^2) \partial_\phi^2 \tag{1}$$

Here $Q$ is the (modified) Carter operator:

$$Q = \left( \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta \right) + \cot^2 \theta \partial_\phi^2 + a^2 \sin^2 \theta \partial_t^2$$

See from this: $\partial_t, \partial_\phi, Q$ commute with $\Box$

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# Symmetries and hidden symmetries

- A Killing field $\xi$ is a symmetry of the wave operator $[\mathcal{L}_\xi, \Box_g] = 0$
- Minkowski: Poincaré Lie algebra $\mathfrak{so}(3,1) \ltimes \mathbf{R}^4$, conformal symmetries (dilation, $\mathbf{K}$)
- Schwarzschild: $\partial_t$, $\mathfrak{so}(3)$ — conserved quantities $\mathcal{E}, \mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z$
- Kerr: $\partial_t, \partial_\phi$ — conserved quantities $\mathcal{E}, \mathcal{L}_z, \mathcal{Q}$
- The Carter constant $\mathcal{Q} = Q_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta$ does not correspond to a Killing field
- The presence of $\mathcal{Q}$ allows the geodesic equations on Kerr to be separated.
- The second order operator $Q = \nabla_\alpha Q^{\alpha\beta}\nabla_\beta$ commutes with the wave operator $[Q, \Box_g] = 0 \leftrightarrow$ hidden symmetry

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# Remarks

- Kerr admits a Killing tensor $Q_{\alpha\beta} = Q_{(\alpha\beta)}$, $\nabla_{(\alpha}Q_{\beta\gamma)} = 0$.
- $Q_{\alpha\beta}$ is related to
  - Killing-Yano 2-form $K_{\alpha\beta} = K_{[\alpha,\beta]}$, $\nabla_{(\alpha}K_{\beta)\gamma} = 0$
  - Killing spinor $K_{AB}$, $\nabla_{A(A'}K_{BC)} = 0$.
- $Q, K$ are related to symmetry operators:
  - $Q = \nabla_\alpha Q^{\alpha\beta}\nabla_\beta$ with $[\Box_g, Q] = 0$,
  - $K = i\gamma_5\gamma^\mu(K_\mu{}^\nu\nabla_\nu - \frac{1}{6}\gamma^\nu\gamma^\lambda\nabla_\lambda K_{\mu\nu})$, with $[D, K]_+ = 0$
- For the spin-$s$ field equation (eg. $s = 1$ Maxwell, $s = 2$ Linearized Einstein), the *extreme* spin scalars solve the spin-$s$ equation

$$\Box_s\psi^{(s)} = \left[(\nabla^\mu + s\Gamma^\mu)(\nabla_\mu + s\Gamma_\mu) + s^2\Psi_2^A\right]\psi^{(s)}$$

  (here $\Gamma^\mu$ is a certain “connection vector”).
- $(\Sigma\Box_s)\psi^{(s)} = 4\pi\Sigma T$ is the Teukolsky Master Equation (TME$_s$) for spin $s$. In particular $\Box u = 0$ is precisely the $s = 0$ vacuum TME.

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# Remarks

- Teukolsky (Teukolsky, 1972) noted that the Carter constant allows to separate the TME into *radial* and *angular* equations for a *separated wave form* $\psi(t,r,\theta,\phi) = e^{-i\omega t}e^{im\phi}R(r)Y(\theta)$
- The $\text{TME}_s$ has a *long-range potential* → difficult to analyze
- Whiting (Whiting, 1989) proved boundedness for the separated wave forms.
- For Maxwell on Schwarzschild, the spin-zero component $\phi_1$ solves the wave equation

$$\Box_g(r\phi_1) + \frac{2M}{r^3}(r\phi_1) = 0$$

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# Canonical analysis

- Let $\mu = \sin\theta$. Then $\sqrt{-g} = \Sigma\mu$, and

$$\Box = \frac{\Sigma}{\sqrt{-g}}\partial_\alpha\left(g^{\alpha\beta}\Sigma\frac{\sqrt{-g}}{\Sigma}\partial_\beta\right) = \frac{1}{\mu}\partial_\alpha(\mathcal{G}^{\alpha\beta}\mu\partial_\beta)$$

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- $\Box u = 0$ is the E-L equation for $S = \int \mathcal{L} d\mu = \frac{1}{2}\int \mathcal{G}^{\alpha\beta}u_\alpha u_\beta d\mu$
- Canonical energy-momentum tensor:

$$\mathcal{T}^\alpha{}_\beta = \frac{\partial\mathcal{L}}{\partial(u_\alpha)}u_\beta - \mathcal{L}\delta^\alpha{}_\beta = u^\alpha u_\beta - \frac{1}{2}u^\gamma u_\gamma \delta^\alpha{}_\beta$$

- Momentum vector:

$$P^\alpha = \mathcal{T}^\alpha{}_\beta X^\alpha + qu^\alpha u - q^\alpha u^2/2$$

  ($q$-terms control $u^\gamma u_\gamma$ in bulk $\partial_\alpha \mu P^\alpha$ )

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$$\mathcal{T}^\alpha{}_\beta = \frac{\partial\mathcal{L}}{\partial(u_\alpha)}u_\beta - \mathcal{L}\delta^\alpha{}_\beta = u^\alpha u_\beta - \frac{1}{2}u^\gamma u_\gamma\delta^\alpha{}_\beta$$

- Momentum vector:

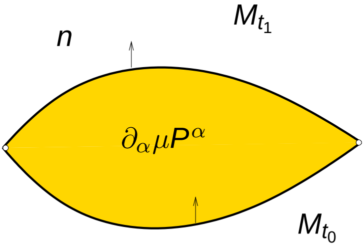
$$P^\alpha = \mathcal{T}^\alpha{}_\beta X^\alpha + qu^\alpha u - q^\alpha u^2/2$$

($q$-terms control $u^\gamma u_\gamma$ in bulk $\partial_\alpha \mu P^\alpha$ )

# Canonical analysis

- Energy:

$$E(t) = -\int_{M_t} P^0 d\mu$$



- Conservation law:

$$E(t_1) - E(t_0) = -\int_{M\times[t_0,t_1]} \partial_\alpha(\mu P^\alpha)$$

- Bulk:

$$\frac{1}{\mu}\partial_\alpha(\mu P^\alpha) = \Box u u_\beta \xi^\beta + \mathcal{T}^\mu{}_\nu \xi^\nu{}_\mu - \frac{1}{2}\frac{1}{\mu}\xi^\gamma \partial_\gamma(\mu \mathcal{G}^{\alpha\beta}) u_\alpha u_\beta$$
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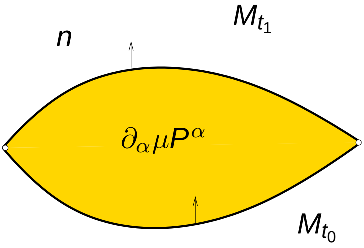

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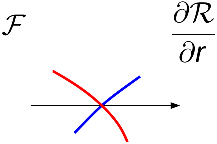
# Trapping in Kerr

- $\mathcal{R}$ is the potential for null geodesics
- The photon sphere is given by
  $$\mathcal{R} = 0, \quad \frac{\partial \mathcal{R}}{\partial r} = 0$$



- Bulk term for **A** must be positive. Contains terms of the form $\mathcal{F}\mathcal{R}'$, need $-\mathcal{F}\frac{\partial \mathcal{R}}{\partial r} \geq 0 \Rightarrow \mathcal{F}$ changes sign at photon orbits
- But $\mathcal{R} = \mathcal{R}(r, \mathcal{E}, \mathcal{L}_z, \mathcal{Q}) \Rightarrow$

  $$\underline{\text{location of photon orbits depends on } \mathcal{E}, \mathcal{L}_z, \mathcal{Q}}$$

- $\Rightarrow$ **A** must be generalized vector field

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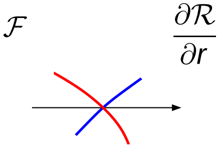

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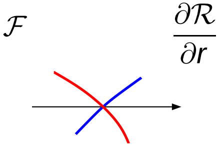

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# Generalized momentum

- $\mathbb{S}_2 = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, Q\}$ — 2nd order symmetry operators.
- If $\Box u = 0$, then $\Box S_{\underline{a}} u = 0$ for $S_{\underline{a}} \in \mathbb{S}_2$
- Given $\mathcal{T}[u]^\alpha{}_\beta$, by polarization define $\mathcal{T}[u,v]^\alpha{}_\beta$.
- Generalized vector field $\xi^{\underline{ab}\beta} = \xi^{(\underline{ab})\beta}$
- Generalized momentum

$$P^\alpha[u,\xi] = \mathcal{T}^\alpha{}_\beta[S_{\underline{a}}u, S_{\underline{b}}u]\xi^{\underline{ab}\beta} + q^{\underline{ab}}(S_{\underline{a}}u)^\alpha(S_{\underline{b}}u) - \frac{1}{2}(q^{\underline{ab}})^\alpha(S_{\underline{a}}u)(S_{\underline{b}}u)$$

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# Outline

1. Background
2. Hidden symmetries
3. Cut and paste energy
4. Generalized Morawetz
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6. Concluding remarks

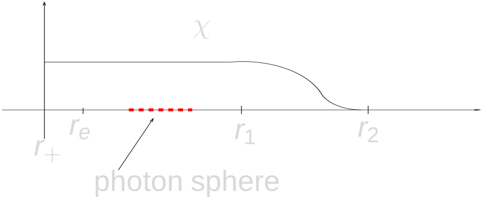
# Cut and paste energy

- $\partial_t$ fails to be timelike in ergoregion $r_+ \leq r < r_e$.
- Use a time-like "almost Killing field"

$$T_\chi = \partial_t + \chi \partial_\phi$$

where $\chi$ is a cutoff function. Want $T_\chi$ is timelike for $r > r_+$.

- Choose $\chi$ so that $\chi(r_+) = \omega_B = \frac{a}{r_+^2 + a^2}$ and supp $\chi' \subset [r_1, r_2]$, $\chi(r_2) = 0$. Choose $[r_1, r_2]$ well outside the photon sphere.



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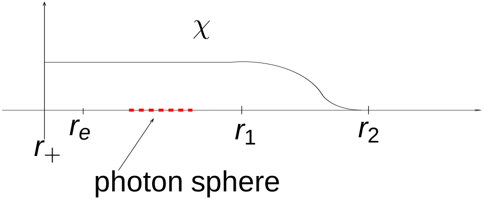

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- Aim for
$$|E_{T_\chi}(t_1) - E_{T_\chi}(t_0)| \leq Ca(E_{T_\chi}(t_1) + E_{T_\chi}(t_0))$$
this gives
$$E_{T_\chi}(t_1) \lesssim \left(\frac{1+Ca}{1-Ca}\right) E_{T_\chi}(t_0)$$
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# Generalized **A**

- $\mathbf{A} = S_{\underline{a}}\mathcal{F}^{\underline{ab}}\partial_r S_{\underline{b}}, \quad S_{\underline{a}} \in \mathbb{S}_2$
- $q = \mathcal{F}'/2 - \frac{1}{2}\mathcal{F}z^{-1}z'$
- If $\Box u = 0$, for suitably chosen $\mathcal{F}$,

$$\frac{1}{\mu}(\partial_\alpha \mu P^\alpha) = \mathcal{A}^{\underline{ab}}(S_{\underline{a}}u)'(S_{\underline{b}}u)' + \mathcal{U}^{\underline{ab}\alpha\beta}(S_{\underline{a}}u)_\alpha(S_{\underline{b}}u)_\beta + \mathcal{V}^{\underline{ab}}(S_{\underline{a}}u)(S_{\underline{b}}u)$$

where

$$\mathcal{A}^{\underline{ab}} = \mathcal{A}^{(\underline{a}}\mathcal{L}^{\underline{b})}$$

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# Generalized **A**

- The quadratic form term $\mathcal{U}^{\underline{ab}\alpha\beta}(S_{\underline{a}}u)_\alpha(S_{\underline{b}}u)_\beta$ is explicitly non-negative modulo timelike boundary terms $B$.
- Positivity of the **A** bulk reduces for small $a$ to the Hardy estimate

$$\int_{r_+}^{\infty}(Au'^2 + Vu^2)dr \geq 0$$

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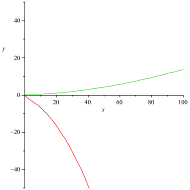
## Generalized **A**

- After substitutions $u = vA^{-1/2}$, $r = r_+ + x$, and setting $M = 1$, this is equivalent to asking whether the equation

  $$v'' + \frac{9x^2 - 34x - 2}{6x^2(x+2)^2} v = 0$$

  has a positive solution on $x > 0$. This is Gauss' hypergeometric equation of the first type!



# Generalized **A**

Solutions are

$$v_1 = (x+2)^{-\frac{3}{2}\sqrt{2}+\frac{1}{2}} x^{\frac{1}{2}+\frac{1}{6}\sqrt{2}\sqrt{3}} {}_2F_1([n_{1,1}, n_{1,2}], [d_{1,1}], -x/2)$$

$$v_2 = (x+2)^{-\frac{3}{2}\sqrt{2}+\frac{1}{2}} x^{\frac{1}{2}-\frac{1}{6}\sqrt{2}\sqrt{3}} {}_2F_1([n_{2,1}, n_{2,2}], [d_{2,1}], -x/2)$$

where ${}_2F_1$ is Gauss' hypergeometric function, and

$$n_{1,1} = \frac{1}{6}(-9+\sqrt{3})\sqrt{2} + \frac{1}{2} + \frac{1}{2}\sqrt{7},$$

$$n_{1,2} = \frac{1}{6}(-9+\sqrt{3})\sqrt{2} + \frac{1}{2} - \frac{1}{2}\sqrt{7}$$

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One shows $v_1$ is positive.

# Remarks

- Necessary to work with higher order energies $E_{T_\chi}[S_{\underline{a}} \cdots S_{\underline{b}} u]$
- $L_z = 0$ mode needs special treatement
- Boundary terms from **A** bulk contain lower order terms (involving $q$) which need to be handled using a Hardy type estimate.
- Once we have the trapping estimate, the dispersive estimate (decay) can be done analogously to the Schwarzschild case.
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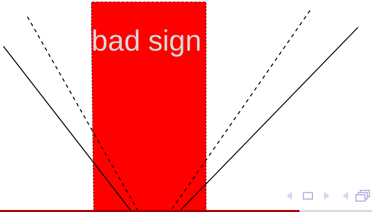
# K vector field

- Tortoise coordinate $dx = \frac{(r^2+a^2)}{\Delta} dr, \quad x|_{r=3M} = 0$
- **K** vector field

$$\mathbf{K} = \frac{1}{2}(t^2 + x^2 + 1)T_\perp + tx\widetilde{N}^2\partial_x$$

where $T_\perp \cong \partial_t + \omega\partial_\phi$, $\omega = \frac{2aMr}{\Pi}$, $\widetilde{N}^2 = \frac{(r^2+a^2)^2}{\Pi}$

- Conjugated wave $\psi = (r^2+a^2)^{1/2}u$
- **K** bulk term has bad sign in a region $r_0 \le r \le r_1$
- Cutoff, higher order Morawetz $\mathbf{A}_2 = t^2\chi(t/x)\mathbf{A}$ controls **K** bulk



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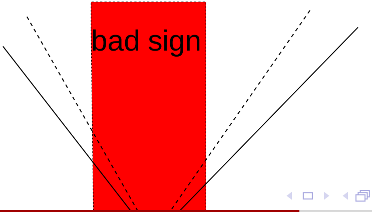

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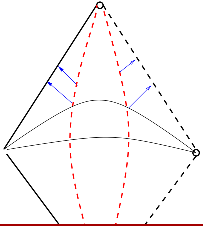
- In Kerr we have a loss, get

$$\text{local energy } \sim t^{-1+Ca}$$

- To get decay at $\mathcal{J}$ and the horizon, integrate these estimates along horizon and $\mathcal{J}$ penetrating foliations to show that local decay implies global decay.



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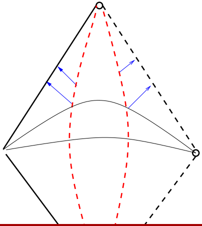

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- For near horizon decay, use coordinates with correct angular behavior up to horizon: $(t, x, \theta, \varphi)$, with $\varphi = \phi - \omega_{\mathcal{H}} t$
- almost null coordinates $v_\pm = t \pm x$
- Conjugated wave operator $-4\partial_+\partial_- + \frac{\Delta}{(r^2+a^2)}\mathcal{Z}$
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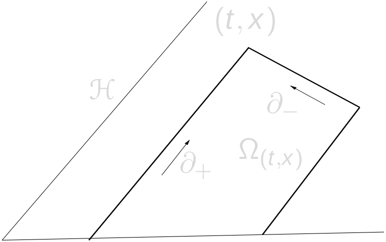

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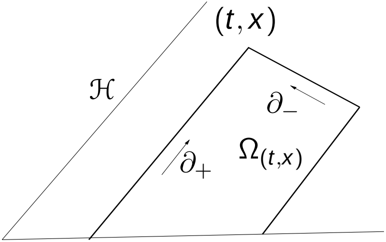

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- For far decay, $r > 3M$, use **K**-energy on almost null slicing

$$v_\pm = t \pm y$$

with $dy = h dx$, $h \sim 1 - C/r^2$

- Final form of the estimate (as in Schwartzschild, with loss),

$$
\begin{aligned}
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|u| &\lesssim I \frac{\langle v_- \rangle^{Ca}}{r} \left( \frac{v_+ - |v_-| + 1}{v_+(1 + |v_-|)} \right)^{1/2}, && \text{for } r > 3M \\
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# References I

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