

# Dirac type operators on Lorentzian manifolds

Christian Bär

joint work with

Nicolas Ginoux and Frank Pfäffle

Institut für Mathematik  
Universität Potsdam

Greifswald, 21. July 2009



# Outline

- 1 Wave Equations
- 2 Quantization
- 3 Dirac type operators



# Wave Operators

Throughout let  $M$  denote a timeoriented Lorentzian manifold.  
Let  $E \rightarrow M$  be a vector bundle.  
Denote the smooth sections in  $E$  by  $C^\infty(M, E)$ .

## Definition

A **wave operator** or **normally hyperbolic operator** is a linear differential operator  $P : C^\infty(M, E) \rightarrow C^\infty(M, E)$  of second order which looks locally like

$$P = - \sum_{i,j=1}^n g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^n A_j(x) \frac{\partial}{\partial x^j} + B(x)$$



# Wave Operators; Examples

- d'Alembert operator (functions)

$$P = \square$$

- Klein-Gordon operator (functions)

$$P = \square + m^2 \text{ or } P = \square + m^2 + \kappa \cdot \text{scal}$$

- Wave operator in electro-dynamics (1-forms)

$$P = d\delta + \delta d$$

- Square of Dirac operator (spinors)

$$P = -D^2$$



# Cauchy Problem

Let  $M$  be globally hyperbolic and let  $S \subset M$  be a smooth spacelike Cauchy hypersurface. Let  $\nu$  be the future directed timelike unit normal field along  $S$ .

## Theorem

For each  $u_0, u_1 \in C_c^\infty(S, E)$  and for each  $f \in C_c^\infty(M, E)$  there exists a unique  $u \in C^\infty(M, E)$  satisfying

$$\begin{cases} Pu = f, & \text{on } M \\ u|_S = u_0, & \text{along } S \\ \nabla_\nu u = u_1, & \text{along } S \end{cases}$$



# Cauchy Problem

## Well-posedness

The solution  $u$  depends continuously on the data  $f$ ,  $u_0$ , and  $u_1$ .

## Finite propagation speed

Moreover,

$$\text{supp}(u) \subset J_+^M(K) \cup J_-^M(K)$$

where  $K =$

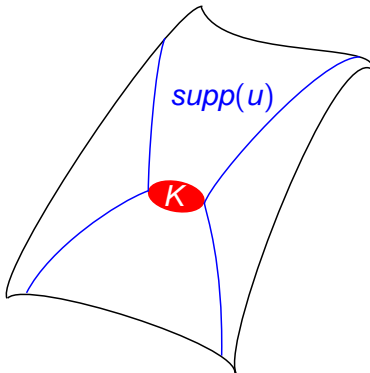
$$\text{supp}(u_0) \cup \text{supp}(u_1) \cup \text{supp}(f).$$



# Cauchy Problem

## Well-posedness

The solution  $u$  depends continuously on the data  $f$ ,  $u_0$ , and  $u_1$ .



## Finite propagation speed

Moreover,

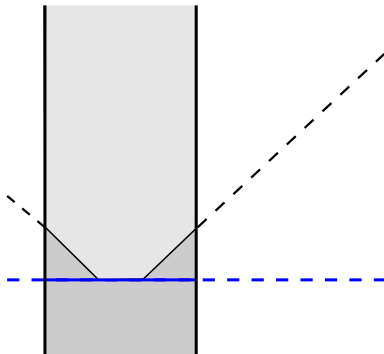
$$\text{supp}(u) \subset J_+^M(K) \cup J_-^M(K)$$

where  $K =$

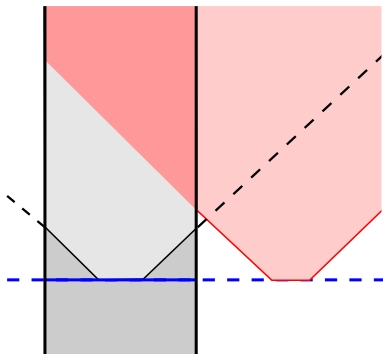
$$\text{supp}(u_0) \cup \text{supp}(u_1) \cup \text{supp}(f).$$



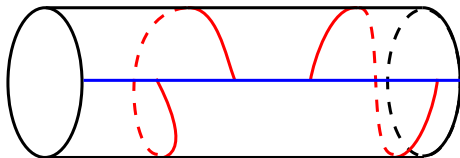
# Cauchy Problem; What Can Go Wrong



# Cauchy Problem; What Can Go Wrong



# Cauchy Problem; What Can Go Wrong



# Green's Operators

## Definition

A linear operator  $G : C_c^\infty(M, E) \rightarrow C^\infty(M, E)$  is called a **Green's operator** for  $P$  if

$$P \circ G = G \circ P = \text{id}_{C_c^\infty(M, E)}$$

## Definition

A Green's operator  $G$  is called **advanced** or **retarded** resp. if

$$\text{supp}(G(u)) \subset J_+(\text{supp}(u)) \text{ or } J_-(\text{supp}(u))$$

resp. for any  $u \in C_c^\infty(M, E)$ .



# Green's Operators

## Definition

A linear operator  $G : C_c^\infty(M, E) \rightarrow C^\infty(M, E)$  is called a **Green's operator** for  $P$  if

$$P \circ G = G \circ P = \text{id}_{C_c^\infty(M, E)}$$

## Definition

A Green's operator  $G$  is called **advanced** or **retarded** resp. if

$$\text{supp}(G(u)) \subset J_+(\text{supp}(u)) \text{ or } J_-(\text{supp}(u))$$

resp. for any  $u \in C_c^\infty(M, E)$ .



# Green's Operators

## Theorem

Let  $P$  be a wave operator over a globally hyperbolic manifold  $M$ .

Then there exist unique advanced and retarded Green's operators for  $P$ .

*These Green's operators are continuous.*

*The sequence of linear maps*

$$0 \rightarrow C_c^\infty(M, E) \xrightarrow{P} C_c^\infty(M, E) \xrightarrow{G_+ - G_-} C_{sc}^\infty(M, E) \xrightarrow{P} C_{sc}^\infty(M, E)$$

*is exact.*



# Green's Operators

## Theorem

Let  $P$  be a wave operator over a globally hyperbolic manifold  $M$ .

Then there exist unique advanced and retarded Green's operators for  $P$ .

These Green's operators are continuous.

*The sequence of linear maps*

$$0 \rightarrow C_c^\infty(M, E) \xrightarrow{P} C_c^\infty(M, E) \xrightarrow{G_+ - G_-} C_{sc}^\infty(M, E) \xrightarrow{P} C_{sc}^\infty(M, E)$$

is exact.



# Green's Operators

## Theorem

Let  $P$  be a wave operator over a globally hyperbolic manifold  $M$ .

Then there exist unique advanced and retarded Green's operators for  $P$ .

These Green's operators are continuous.

The sequence of linear maps

$$0 \rightarrow C_c^\infty(M, E) \xrightarrow{P} C_c^\infty(M, E) \xrightarrow{G_+ - G_-} C_{sc}^\infty(M, E) \xrightarrow{P} C_{sc}^\infty(M, E)$$

is exact.



# Outline

- 1 Wave Equations
- 2 Quantization
- 3 Dirac type operators



# Fock Space

$H$  complex Hilbert space,  $\odot^n H$  completion of  $\odot_{\text{alg}}^n H$

(Bosonic or symmetric) Fock space  $\mathfrak{F}(H)$  is the completion of

$$\mathfrak{F}_{\text{alg}}(H) := \bigoplus_{n=0}^{\infty} \odot^n H.$$

Fix  $v \in H$ . Define the creation operator

$$a^*(v)v_1 \odot \dots \odot v_n := v \odot v_1 \odot \dots \odot v_n$$

and the annihilation operator

$$a(v)(w_0 \odot \dots \odot w_n) := \sum_{k=0}^n (v, w_k) w_0 \odot \dots \odot \hat{w}_k \odot \dots \odot w_n$$



# Fock Space

$H$  complex Hilbert space,  $\bigcirc^n H$  completion of  $\bigcirc_{\text{alg}}^n H$

(Bosonic or symmetric) Fock space  $\mathfrak{F}(H)$  is the completion of

$$\mathfrak{F}_{\text{alg}}(H) := \bigoplus_{n=0}^{\infty} \bigcirc^n H.$$

Fix  $v \in H$ . Define the **creation operator**

$$a^*(v)v_1 \odot \dots \odot v_n := v \odot v_1 \odot \dots \odot v_n$$

and the **annihilation operator**

$$a(v)(w_0 \odot \dots \odot w_n) := \sum_{k=0}^n (v, w_k) w_0 \odot \dots \odot \hat{w}_k \odot \dots \odot w_n$$



# Canonical Commutator Relations

Canonical commutator relations:

$$[a(v), a(w)] = [a^*(v), a^*(w)] = 0,$$

$$[a(v), a^*(w)] = (v, w) \cdot \text{id}.$$

Definition

Segal operator:

$$\theta(v) := \frac{1}{\sqrt{2}}(a(v) + a^*(v))$$

The Segal operator on  $\mathfrak{F}_{\text{alg}}(H)$  is essentially self-adjoint in  $\mathfrak{F}(H)$ .

$$[\theta(v), \theta(w)] = i \cdot \mathfrak{Im}(v, w)$$



# Canonical Commutator Relations

Canonical commutator relations:

$$[a(v), a(w)] = [a^*(v), a^*(w)] = 0,$$

$$[a(v), a^*(w)] = (v, w) \cdot \text{id}.$$

Definition

Segal operator:

$$\theta(v) := \frac{1}{\sqrt{2}}(a(v) + a^*(v))$$

The Segal operator on  $\mathfrak{F}_{\text{alg}}(H)$  is essentially self-adjoint in  $\mathfrak{F}(H)$ .

$$[\theta(v), \theta(w)] = i \cdot \mathfrak{Im}(v, w)$$



# Geometric Setup

- Globally hyperbolic Lorentzian manifold  $M$
- **Real** vector bundle  $E \rightarrow M$  with non-degenerate metric
- Formally selfadjoint wave operator  $P$  on  $E$

## Definition

A **twist structure of spin  $k/2$**  on  $E$  is a smooth section  $Q \in C^\infty(M, \text{Hom}(\odot^k TM, \text{End}(E)))$  such that:

- $\langle Q(X_1 \odot \cdots \odot X_k)e, f \rangle = \langle e, Q(X_1 \odot \cdots \odot X_k)f \rangle$
- If  $X$  is future directed timelike, then the bilinear form  $\langle \cdot, \cdot \rangle_X$  defined by

$$\langle f, g \rangle_X := \langle Q_X f, g \rangle$$

is positive definite where  $Q_X = Q(X \odot \cdots \odot X)$ .



# Geometric Setup

- Globally hyperbolic Lorentzian manifold  $M$
- **Real** vector bundle  $E \rightarrow M$  with non-degenerate metric
- Formally selfadjoint wave operator  $P$  on  $E$

## Definition

A **twist structure of spin  $k/2$**  on  $E$  is a smooth section  $Q \in C^\infty(M, \text{Hom}(\odot^k TM, \text{End}(E)))$  such that:

- $\langle Q(X_1 \odot \cdots \odot X_k)e, f \rangle = \langle e, Q(X_1 \odot \cdots \odot X_k)f \rangle$
- If  $X$  is future directed timelike, then the bilinear form  $\langle \cdot, \cdot \rangle_X$  defined by

$$\langle f, g \rangle_X := \langle Q_X f, g \rangle$$

is positive definite where  $Q_X = Q(X \odot \cdots \odot X)$ .



# Examples

## Example

If the metric on  $E$  is positive definite, one can choose  $k = 0$  and  $Q = \text{id}$

## Example

For spinor bundle  $E$  let  $k = 1$  and  $Q(X)$  be Clifford multiplication by  $X$

## Example

For  $E = \Lambda^q T^*M$  let  $k = 2$  and

$$Q(X \odot Y)\alpha := X^\flat \wedge \iota_Y \alpha + Y^\flat \wedge \iota_X \alpha - \langle X, Y \rangle \cdot \alpha$$



# Examples

## Example

If the metric on  $E$  is positive definite, one can choose  $k = 0$  and  $Q = \text{id}$

## Example

For **spinor bundle**  $E$  let  $k = 1$  and  $Q(X)$  be **Clifford multiplication** by  $X$

## Example

For  $E = \Lambda^q T^*M$  let  $k = 2$  and

$$Q(X \odot Y)\alpha := X^b \wedge \iota_Y \alpha + Y^b \wedge \iota_X \alpha - \langle X, Y \rangle \cdot \alpha$$



# Examples

## Example

If the metric on  $E$  is positive definite, one can choose  $k = 0$  and  $Q = \text{id}$

## Example

For **spinor bundle**  $E$  let  $k = 1$  and  $Q(X)$  be **Clifford multiplication** by  $X$

## Example

For  $E = \Lambda^q T^*M$  let  $k = 2$  and

$$Q(X \odot Y)\alpha := X^b \wedge \iota_Y \alpha + Y^b \wedge \iota_X \alpha - \langle X, Y \rangle \cdot \alpha$$



# Geometric Setup

- Globally hyperbolic Lorentzian manifold  $M$
- Real vector bundle  $E \rightarrow M$  with non-degenerate metric
- Formally selfadjoint wave operator  $P$  on  $E$
- Twist structure  $Q$
- Cauchy hypersurface  $S \subset M$

We get real Hilbert space  $L^2(S, E^*)$  where

$$(u, v)_S := \int_S \langle u, v \rangle_\nu dA = \int_S \langle Q_\nu^* u, v \rangle dA$$



# Geometric Setup

- Globally hyperbolic Lorentzian manifold  $M$
- Real vector bundle  $E \rightarrow M$  with non-degenerate metric
- Formally selfadjoint wave operator  $P$  on  $E$
- Twist structure  $Q$
- Cauchy hypersurface  $S \subset M$

We get real Hilbert space  $L^2(S, E^*)$  where

$$(u, v)_S := \int_S \langle u, v \rangle_\nu dA = \int_S \langle Q_\nu^* u, v \rangle dA$$



# Quantum Field

- Apply Fock space construction to  $H_S := L^2(S, E^*) \otimes_{\mathbb{R}} \mathbb{C}$
- Get Segal field  $\theta$

## Definition

**Quantum field:** For  $f \in C_c^\infty(M, E^*)$  put

$$\Phi_S(f) := \theta(\underbrace{i(G_+^* - G_-^*)f|_S - (Q_\nu^*)^{-1} \nabla_\nu((G_+^* - G_-^*)f)}_{\in H_S}).$$



# Haag-Kastler Axioms

## Theorem

- $C_c^\infty(M, E^*) \rightarrow \mathfrak{F}(H_S)$ ,  $f \mapsto \Phi_S(f)\omega$ , is continuous for any  $\omega \in \mathfrak{F}_{\text{alg}}(H_S)$
- $P\Phi_S = 0$  in the distributional sense
- $[\Phi_S(f), \Phi_S(g)] = 0$  if the supports of  $f$  and  $g$  are causally independent.
- The linear span of the vectors  $\Phi_S(f_1) \cdots \Phi_S(f_n)\Omega$  is dense in  $\mathfrak{F}(H_S)$  where  $\Omega = 1 \in \odot^0 H_S = \mathbb{C}$  is the vacuum vector.



# Haag-Kastler Axioms

## Theorem

- $C_c^\infty(M, E^*) \rightarrow \mathfrak{F}(H_S)$ ,  $f \mapsto \Phi_S(f)\omega$ , is continuous for any  $\omega \in \mathfrak{F}_{\text{alg}}(H_S)$
- $P\Phi_S = 0$  in the distributional sense
- $[\Phi_S(f), \Phi_S(g)] = 0$  if the supports of  $f$  and  $g$  are causally independent.
- The linear span of the vectors  $\Phi_S(f_1) \cdots \Phi_S(f_n)\Omega$  is dense in  $\mathfrak{F}(H_S)$  where  $\Omega = 1 \in \odot^0 H_S = \mathbb{C}$  is the vacuum vector.



# Haag-Kastler Axioms

## Theorem

- $C_c^\infty(M, E^*) \rightarrow \mathfrak{F}(H_S)$ ,  $f \mapsto \Phi_S(f)\omega$ , is continuous for any  $\omega \in \mathfrak{F}_{\text{alg}}(H_S)$
- $P\Phi_S = 0$  in the distributional sense
- $[\Phi_S(f), \Phi_S(g)] = 0$  if the supports of  $f$  and  $g$  are causally independent.
- The linear span of the vectors  $\Phi_S(f_1) \cdots \Phi_S(f_n)\Omega$  is dense in  $\mathfrak{F}(H_S)$  where  $\Omega = 1 \in \odot^0 H_S = \mathbb{C}$  is the vacuum vector.



# Haag-Kastler Axioms

## Theorem

- $C_c^\infty(M, E^*) \rightarrow \mathfrak{F}(H_S)$ ,  $f \mapsto \Phi_S(f)\omega$ , is continuous for any  $\omega \in \mathfrak{F}_{\text{alg}}(H_S)$
- $P\Phi_S = 0$  in the distributional sense
- $[\Phi_S(f), \Phi_S(g)] = 0$  if the supports of  $f$  and  $g$  are causally independent.
- The linear span of the vectors  $\Phi_S(f_1) \cdots \Phi_S(f_n)\Omega$  is dense in  $\mathfrak{F}(H_S)$  where  $\Omega = 1 \in \odot^0 H_S = \mathbb{C}$  is the **vacuum vector**.



# Problems

## Problems:

- Construction depends on choice of Cauchy hypersurface
- Microlocal spectrum condition is violated

## Algebraic quantum field theory:

- Forget Fock space (and particles)
- Regard observables (operators) as primary objects
- To each (reasonable) spacetime region associate an algebra of observables



# Problems

Problems:

- Construction depends on choice of Cauchy hypersurface
- Microlocal spectrum condition is violated

Algebraic quantum field theory:

- Forget Fock space (and particles)
- Regard observables (operators) as primary objects
- To each (reasonable) spacetime region associate an algebra of observables



# CCR-algebras

Let  $(V, \omega)$  be a symplectic vector space.

## Definition

A **CCR-algebra** of  $(V, \omega)$  consists of a  $C^*$ -algebra  $A$  with unit and a map  $W : V \rightarrow A$  such that for all  $\phi, \psi \in V$  we have

- $W(0) = 1$
- $W(-\phi) = W(\phi)^*$
- $W(\phi) \cdot W(\psi) = e^{-i\omega(\phi, \psi)/2} W(\phi + \psi)$
- $A$  is generated by the  $W(\phi)$

## Theorem

*To each symplectic vector space there exists a CCR-algebra, unique up to  $*$ -isomorphism.*



# CCR-algebras

Let  $(V, \omega)$  be a symplectic vector space.

## Definition

A **CCR-algebra** of  $(V, \omega)$  consists of a  $C^*$ -algebra  $A$  with unit and a map  $W : V \rightarrow A$  such that for all  $\phi, \psi \in V$  we have

- $W(0) = 1$
- $W(-\phi) = W(\phi)^*$
- $W(\phi) \cdot W(\psi) = e^{-i\omega(\phi, \psi)/2} W(\phi + \psi)$
- $A$  is generated by the  $W(\phi)$

## Theorem

*To each symplectic vector space there exists a CCR-algebra, unique up to  $*$ -isomorphism.*



# Construction of the Symplectic Vector Space

Let  $M$  be globally hyperbolic, let  $P$  a formally self-adjoint wave operator acting on sections in  $E$ .

Let  $G_+$  and  $G_-$  be the Green's operators of  $P$ .

$$\tilde{\omega}(\phi, \psi) := \int_M \langle (G_+ - G_-)\phi, \psi \rangle dVol$$

defines a degenerate symplectic form on  $C_c^\infty(M, E)$ .  
It induces a (nondegenerate) symplectic form  $\omega$  on

$$\begin{aligned} V(M, E, P) &:= C_c^\infty(M, E) / \ker(G_+ - G_-) \\ &\cong \ker(P : C_{sc}^\infty(M, E) \rightarrow C_{sc}^\infty(M, E)) \end{aligned}$$



# Quantization Functor

$\mathfrak{A}_M := CCR(M, E, P) := CCR(V(M, E, P), \omega)$  defines a functor

globally hyperbolic  
manifolds equipped  
with a formally  
self-adjoint wave operator



$C^*$ -algebras  
with unit



# Haag-Kastler Axioms, II

## Theorem

- If  $O_1 \subset O_2$ , then  $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$  for all  $O_1, O_2 \in I$ .
- $\mathfrak{A}_M = \overline{\bigcup_{\substack{O \in I \\ O \neq \emptyset, O \neq M}} \mathfrak{A}_O}$ .
- $\mathfrak{A}_M$  is simple.
- The  $\mathfrak{A}_O$ 's have a common unit 1.
- For all  $O_1, O_2 \in I$  with  $J(\overline{O_1}) \cap \overline{O_2} = \emptyset$  the subalgebras  $\mathfrak{A}_{O_1}$  and  $\mathfrak{A}_{O_2}$  of  $\mathfrak{A}_M$  commute:  $[\mathfrak{A}_{O_1}, \mathfrak{A}_{O_2}] = \{0\}$ .
- **Time-slice axiom.** Let  $O_1 \subset O_2$  be nonempty elements of  $I$  admitting a common Cauchy hypersurface. Then  $\mathfrak{A}_{O_1} = \mathfrak{A}_{O_2}$ .
- Let  $O_1, O_2 \in I$  and let the Cauchy development  $D(O_2)$  be relatively compact in  $M$ . If  $O_1 \subset D(O_2)$ , then  $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$ .



# Comparison of the Two Approaches

Given a Cauchy hypersurface  $S \subset M$ , a twist structure, and the corresponding quantum field  $\Phi_S$

$$W_S(f) := \exp(i\Phi_S(f))$$

defines a CCR-representation for  $V(M, E, P)$ .



# Applications in Physics

- Hawking radiation of black holes
- Unruh effect

Brunetti, Dimock, Fewster, Fredenhagen, Hollands,  
Radzikowski, Verch, Wald, ...



# Outline

- 1 Wave Equations
- 2 Quantization
- 3 Dirac type operators



# Dirac Type Operators

## Definition

A **Dirac type operator** is a linear differential operator  $D : C^\infty(M, E) \rightarrow C^\infty(M, E)$  of first order such that  $P = -D^2$  is normally hyperbolic.

## Examples

- **Classical Dirac operator** (spinors)

$$D = i \sum_{k=1}^n \varepsilon_k e_k \nabla_{e_k}$$

- **Euler operator** (forms)

$$D = i(d - \delta)$$



# Dirac Type Operators

## Definition

A **Dirac type operator** is a linear differential operator  $D : C^\infty(M, E) \rightarrow C^\infty(M, E)$  of first order such that  $P = -D^2$  is normally hyperbolic.

## Examples

- **Classical Dirac operator** (spinors)

$$D = i \sum_{k=1}^n \varepsilon_k \mathbf{e}_k \nabla_{\mathbf{e}_k}$$

- **Euler operator** (forms)

$$D = i(d - \delta)$$



# Analytic Properties

Dirac type operators on globally hyperbolic manifolds inherit good analytical properties from wave operators.

- Cauchy problem is well-posed,

$$\begin{cases} Du = f, & \text{on } M \\ u|_S = u_0, & \text{along } S \end{cases}$$

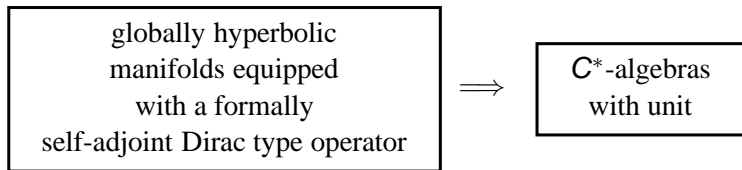
- Existence and uniqueness of advanced and retarded Green's operator.
- Exact sequence

$$0 \rightarrow C_c^\infty(M, E) \xrightarrow{D} C_c^\infty(M, E) \xrightarrow{G_+ - G_-} C_{SC}^\infty(M, E) \xrightarrow{D} C_{SC}^\infty(M, E)$$



# Algebraic CCR-Quantization

Algebraic CCR-quantization can be constructed as for wave operators.



Concept of **twist structures** exists and leads to representation of the CCR-algebras.



# CAR-Quantization?

Physics (Spin-Statistics-Theorem):

Integral spin  $\Rightarrow$  CCR-Quantization

Half-integral spin  $\Rightarrow$  CAR-Quantization

We can construct canonical CAR-quantization

globally hyperbolic  
manifolds equipped  
with a formally  
self-adjoint Dirac type operator  
of definite type



$C^*$ -algebras  
with unit



# CAR-Quantization?

Physics (Spin-Statistics-Theorem):

Integral spin  $\Rightarrow$  CCR-Quantization

Half-integral spin  $\Rightarrow$  CAR-Quantization

We can construct canonical CAR-quantization

globally hyperbolic  
manifolds equipped  
with a formally  
self-adjoint Dirac type operator  
**of definite type**

$\Rightarrow$

$C^*$ -algebras  
with unit

