Causal Variational Principles in Discrete and Continuum Space-Times

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Motivation from Einstein-Dirac system

Begin with a system of Dirac particles in curved space-time (Relativistic Quantum Mechanics + General Relativity)

From the mathematical point of view:

- Lorentzian manifold (M, g), signature $(+ - - -)$

assume that the manifold is spin,

- let SM be the corresponding spinor bundle
  - thus fiber S_xM is 4-dimensional complex vector space, endowed with inner product $\prec .|. \succ$ of signature $(2, 2)$
  - The sections in SM are called wave functions,

$$\Psi \in \Gamma(SM) \;:\; x \mapsto \Psi(x) \in S_xM$$

Motivation from Einstein-Dirac system

- Clifford multiplication:

  $$T_xM \times S_xM \to S_xM \;:\; (u, \Psi) \mapsto u \cdot \Psi$$

  satisfies anti-commutation and symmetry relations

  $$u \cdot v \cdot \Psi + v \cdot u \cdot \Psi = 2\, g(u,v)\, \Psi$$
  $$\prec u \cdot \Psi \,|\, \Phi \succ \;=\; \prec \Psi \,|\, u \cdot \Phi \succ$$

- choose a spin connection

  $$\nabla : T_xM \times \Gamma(SM) \to S_xM \;:\; (u, \Psi) \mapsto \nabla_u \Psi(x)$$

  satisfies Leibniz rules and compatibility conditions
- introduce the Dirac operator by

  $$\mathcal{D} \;:\; \Gamma(SM) \to \Gamma(SM) \;:\; \Psi \to \mathcal{D}\Psi = i \sum_{j=0}^{3} e_j \cdot \nabla_{e_j} \Psi$$

Motivation from Einstein-Dirac system

consider f particles with wave functions $\Psi_1, \ldots, \Psi_f$
(suitably orthonormalized)
Einstein-Dirac equations:

$$\begin{cases} (\mathcal{D} - m)\Psi_\ell = 0 \\ \mathrm{Ric} - \frac{1}{2}\,\mathrm{scal}\, g = 8\pi\kappa\, T \end{cases}$$

on the left: Ricci and scalar curvature
on the right: energy-momentum tensor

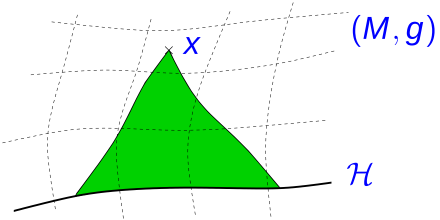
$$T_{jk} = \sum_{\ell=1}^{f} \mathrm{Im} \prec\Psi_\ell \,|\, e_{(j} \nabla_{k)} \Psi_\ell\succ$$

- is a nonlinear system of hyperbolic PDEs
- initial value problem is well-posed

Motivation from Einstein-Dirac system

Causal Structure

- The solution of the initial value problem depends only on the initial data in the causal past



- finite propagation speed, "information cannot travel faster than the speed of light"

Motivation from Einstein-Dirac system

Variational Structure

consider the classical action (Hilbert + Dirac)

$$\mathcal{S} = \int_M \left(-\frac{1}{16\pi\kappa}\, \text{scal} + \sum_{\ell=1}^{f} \prec\Psi_\ell \,|\, (\mathcal{D}-m)\Psi_\ell\succ \right) d\mu_M$$

seek for stationary points of the action

- vary Ψ_ℓ: gives Dirac equations
- vary metric: gives Einstein equations

Works similarly for all other physical equations (electrodynamics, strong and weak interactions, etc.)

- variational principle gives clean unified framework
- unfortunately, many structures must be put in: manifold, metric, connection, spinor bundle, Clifford multiplication, spin connection, . . .

DREAM (≈ 1990)

- formulate more fundamental variational principle which explains at least some of the above structures
- in particular: explain causal structure
- should in a suitable limit give back classical action

this dream has to a large extent come true, as will now be outlined

Local manifestation of causal structure

How is the causal structure encoded locally?

- Consider a tangent space $u \in T_xM$

$$\begin{cases} g(u,u) \geq 0 & \text{if } u \text{ is timelike or null} \\ g(u,u) < 0 & \text{if } u \text{ is spacelike} \end{cases}$$

- Alternatively: consider corresponding Clifford multiplication

$$u : S_xM \to S_xM : \Psi(x) \mapsto u \cdot \Psi(x)$$

is a linear mapping on the spinors and

$$u^2 = u \cdot u = g(u,u)\,\mathrm{id}$$

$$\text{spectrum of } u \text{ is } \begin{cases} \text{real} & \text{if } u \text{ is timelike or null} \\ \text{imaginary} & \text{if } u \text{ is spacelike} \end{cases}$$

Spectral characterization of causal structure

- This concept even works in a neighborhood of x:
  Suppose that P is an operator with integral kernel $P(x,y)$,

  $$(P\Psi)(x) = \int_M P(x,y)\, \Psi(y)\, d\mu_M(y)$$

  (P could be Green's function, spectral projector, …,
  will be specified below)

  $$P(x,y) : S_yM \to S_xM$$

  is not an endomorphism, and its spectrum is not defined
  thus introduce

  $$\text{closed chain} \quad A_{xy} := P(x,y)\, P(y,x) : S_xM \to S_xM$$

Spectral characterization of causal structure

Definition (causal structure)

Space-time points $x, y \in M$ are called

$$\begin{cases} \text{timelike separated} & \text{if } \sigma(A_{xy}) \subset \mathbb{R} \text{ and } \#\sigma(A_{xy}) \neq 1 \\ \text{spacelike separated} & \text{if } \sigma(A_{xy}) \not\subset \mathbb{R} \end{cases}$$

For suitable P, this definition agrees with usual notions, for example

- P is the symmetric Green's function in Minkowski space
- P is the symmetric Green's function in a Lorentzian manifold, agrees locally, i.e. for every x and all y in a neighborhood of x
- P is a spectral projector of the Dirac operator: locally
- P is a Dirac sea configuration: locally

Basic idea for causal variational principles

Formulate a variational principle directly with $P(x,y)$, which is compatible with causal structure

$$\mathcal{S} = \iint_{M\times M} \mathcal{L}[A_{xy}]\, d\mu_M(x)\, d\mu_M(y)$$

and $\mathcal{L}[A_{xy}]$ vanishes for space-like separation

- Standard example, physically most interesting case:

$$\mathcal{L}[A_{xy}] = \frac{1}{8} \sum_{i,j=1}^{4} \left(|\lambda_i| - |\lambda_j|\right)^2 \;\geq\; 0$$

  vanishes if eigenvalues form a complex conjugate pair happens if A_{xy} is symmetric with respect to $\prec .|.\succ$.

The fermionic operator

We need to specify what we want to choose for P.
Idea: Take all particle wave functions,

$$P(x,y) = -\sum_{\ell=1}^{f} \Psi_\ell(x)\,\overline{\Psi_\ell(y)}|$$

Take into account the Dirac sea, i.e. in the Minkowski vacuum

$$P(x,y) = P^{\text{sea}}(x,y) = \int \frac{d^4k}{(2\pi)^4}(\not{k}+m)\,\delta(k^2-m^2)\,\Theta(-k^0)\,e^{-ik(x-y)}$$

or with particles and anti-particles

$$P(x,y) \;=\; P^{\text{sea}}(x,y) + \sum_{k=1}^{n_f} \Psi_k(x)\overline{\Psi_k(y)} - \sum_{l=1}^{n_a} \Phi_l(x)\overline{\Phi_l(y)}$$

Select basic objects and simplify setting

- causal structure generates topology (Alexandrov) and conformal structure
- $P(x,y)$ also yields Lorentzian metric
- $P(x,y)$ contains particle and anti-particle wave functions

Thus $P(x,y)$ describes physical system completely
Try to keep the setting as simple as possible

- finite number f of particles (ultraviolet regularization)
- finite number m of space-time points (infrared reg.)

introduce inner product space $(H, \langle .|. \rangle)$ by

$$\langle \Psi | \Phi \rangle = \sum_{x \in M} \prec \Psi | \Phi \succ(x) \qquad \text{space-time inner product}$$

$$E_x \;:\; H \to S_x M \;:\; \Psi \mapsto \Psi(x) \qquad \text{space-time projectors}$$

Discrete space-time

$(H, \langle .|. \rangle)$ is a finite-dimensional indefinite inner product space
(i.e. $\langle .|. \rangle$ is non-degenerate sesquilinear form)
$M = \{1, \ldots, m\}$ are discrete space-time points,
to every $x \in M$ we associate a *space-time projector* E_x on H:

- projector = symmetric and idempotent,

$$E_x^* \;=\; E_x \;=\; E_x^2$$

- are orthogonal and complete,

$$E_x \, E_y = \delta_{xy} \, E_x \;, \qquad \sum_{x \in M} E_x = \mathbb{1}$$

- The spaces $E_x(H) \subset H$ all have the same signature (n, n).
  n is called spin dimension

$(H, \langle .|. \rangle, (E_x)_{x \in M})$ is called **discrete space-time**

The fermionic operator

Next: introduce **particles** via the fermionic operator P

- $P(H) \subset H$ is f-dimensional and negative definite
- $(-P)$ is positive, i.e. $\langle u | (-P) u \rangle \geq 0 \quad \forall u \in H$

Form composite expressions. Useful notation:

$$\begin{aligned} P(x,y) &= E_x \, P \, E_y \;:\; E_y(H) \rightarrow E_x(H) && \textit{discrete kernel} \\ \Psi(x) &= E_x \, \Psi \;\in\; E_x(H) && \textit{wave function} \end{aligned}$$

We can write the vector $P\Psi$ as follows,

$$(P\Psi)(x) \;=\; E_x \, P \, \Psi \;=\; \sum_{y \in M} E_x \, P \, E_y \, \Psi \;=\; \sum_{y \in M} (E_x \, P \, E_y) \, (E_y \, \Psi)$$

and thus $\quad (P\Psi)(x) \;=\; \sum_{y \in M} P(x,y) \, \Psi(y)$

corresponds to representation with integral kernel.

Setting up a variational principle

$$A_{xy} := P(x,y)\,P(y,x) \;:\; E_x(H) \to E_x(H) \quad \textit{closed chain}$$

denote zeros of its characteristic polynomial by $\lambda_1, \ldots, \lambda_4$

minimize the action $\quad \mathcal{S} = \sum_{x,y \in M} \mathcal{L}[A_{xy}]$

with the Lagrangian $\quad \mathcal{L}[A_{xy}] = \frac{1}{8} \sum_{i,j=1}^{4} \left(|\lambda_i| - |\lambda_j| \right)^2$

trace constraint $\quad \mathrm{Tr}(P) = f$

Is the simplest causal variational in discrete space-time
other variants: fermionic projector $P^2 = P$, other constraints, . . .

Rigorous results

Theorem (*Calc. Var. & PDEs* **29** (2007) 431-453)

The minimum is attained.

surprising result:

Spontaneous breaking of permutation symmetry

discrete space-time and the variational principle are symmetric under permutations of space-time points,

$$\mathcal{S} = \sum_{x,y \in M} \mathcal{L}[A_{xy}] \, .$$

Do minimizers also have this symmetry?

Definition

A subgroup $\mathcal{O} \subset \mathcal{S}_m$ is called *outer symmetry group* if for every $\sigma \in \mathcal{O}$ there is a unitary operator U with

$$UPU^{-1} = P\,, \qquad UE_xU^{-1} = E_{\sigma(x)}\,.$$

Theorem (*Adv. Theor. Math. Phys.* **11** (2007) 91-146)

Assume that

$$m \;>\; \begin{cases} 3 & \text{if } n = 1 \\ \max\left(2n+1,\; 4\,[\log_2 n] + 6\right) & \text{if } n > 1 \end{cases}$$

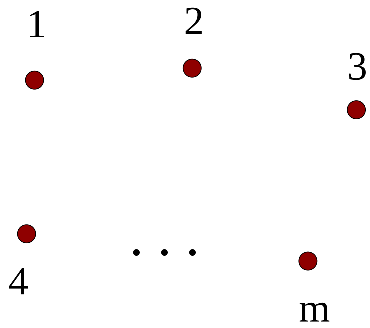
$$n \;<\; f \;<\; m-1\,.$$

Then the discrete fermion system cannot have the outer symmetry group $\mathcal{O} = \mathcal{S}_m$.

covers the physically interesting case $n \ll f \ll m$

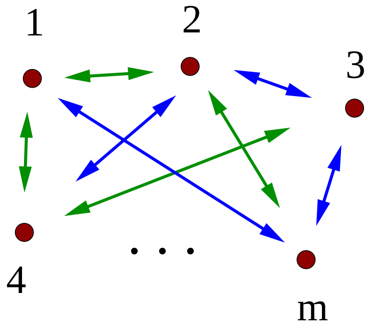
The idea of spontaneous structure formation

begin with finite number of space-time points $M = \{1, \dots, m\}$ without any additional structures



The idea of spontaneous structure formation

- On the space-time points introduce ensemble of wave functions $\Psi_1(x), \ldots, \Psi_f(x)$, $x \in M$
- Set up a variational principle for the wave functions, wave functions organize themselves
- correlations $\Psi_\alpha(x) \leftrightarrow \Psi_\alpha(y)$ yield relations $x \leftrightarrow y$

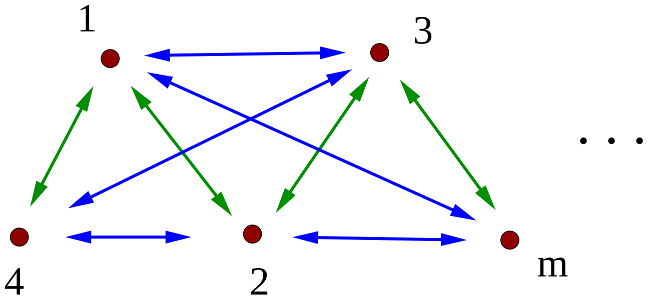


The idea of spontaneous structure formation

- self-organization leads to additional structures

  simplest example: non-trivial outer symmetry group

  more specifically: emergence of space-time structure



discrete causal structure: space-like, time-like, . . .

Simple Numerical Examples

with A. Diethert and D. Schiefeneder (*Int. J. Mod. Phys.* **A** *23:4579-4620 (2008)*)

For simplicity spin dimension $n = 1$,
only two particles ($f = 2$), fermionic projector
choose pseudo-orthonormal basis Ψ_1, Ψ_2 of $P(H)$

$$P = -|\Psi_1\rangle\langle\Psi_1| - |\Psi_2\rangle\langle\Psi_2| \,.$$

Consider for every $x \in M$ the local correlation matrix

$$(F_x)^a_b \;:=\; -\langle \Psi_a \,|\, E_x \Psi_b \rangle \qquad \text{Hermitian } 2 \times 2\text{-matrix}$$

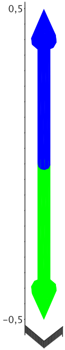
decompose F_x into Pauli matrices:

$$F_x \;=\; \rho_x \, \mathbb{1} \,+\, \vec{v}_x \, \vec{\sigma}$$

gives Bloch vectors $\vec{v}_x \in \mathbb{R}^3$
very useful for visualization of fermion system

Bloch vectors for 2 space-time points

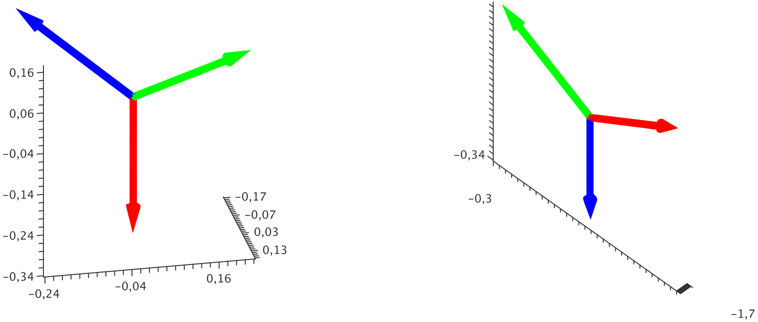
$m = 2$



unique up to rotations in 3-dimensional space

Bloch vectors for 3 space-time points

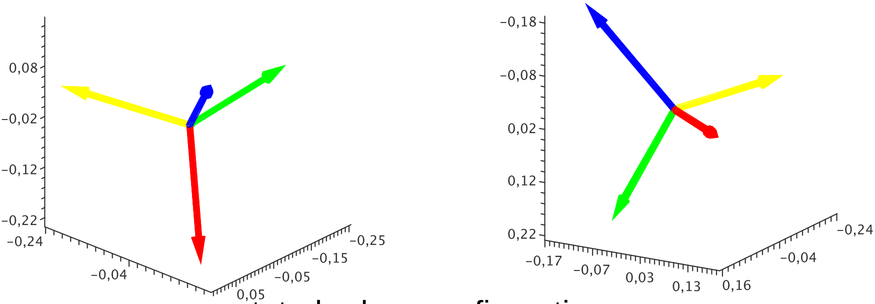
$m = 3$



planar regular triangle, unique up to rotations

Bloch vectors for 4 space-time points

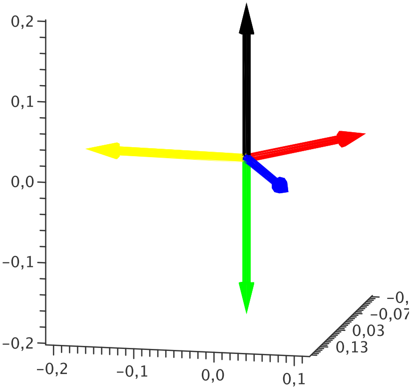
$m = 4, f = 2$



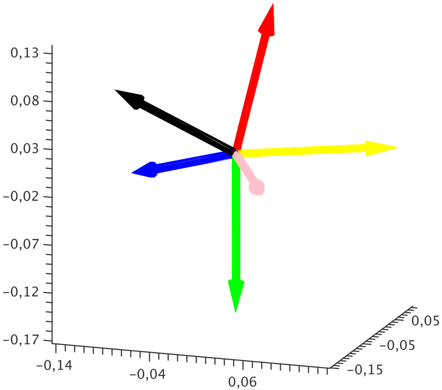
tetrahedron configurations

- are not equivalent, because different orientations
- spontaneous symmetry breaking:
  minimizers distinguish the orientation

Bloch vectors for 5 and 6 space-time points

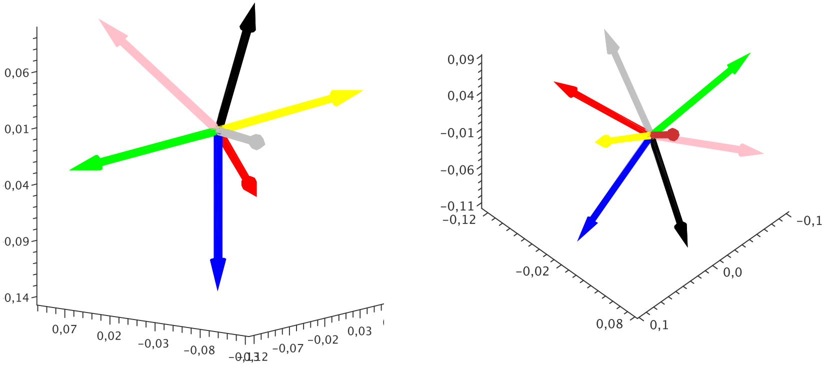


$m = 5$



$m = 6$

Bloch vectors for $m > 6$



$m = 7$ $m = 8$

- spontaneous generation of nearest-neighbor relation
- as $m \to \infty$ structure of 2-dimensional lattice on S^2

A model in the continuum limit

specify vacuum as sum of Dirac seas,

$$P(x,y) = \sum_{\beta=1}^{3} P^{\text{sea}}_{m_\beta}(x,y)$$

$$P^{\text{sea}}_m(x,y) = \int \frac{d^4k}{(2\pi)^4}\,(\not{k}+m)\,\delta(k^2-m^2)\,\Theta(-k^0)\,e^{-ik(x-y)}$$

β labels “generations” of elementary particles

- introduce ultraviolet regularization
- analyze Euler-Lagrange equations as regularization is removed

A model in the continuum limit

Theorem (recent preprint)

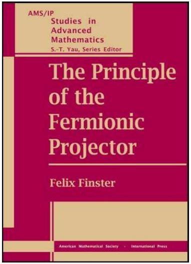
Interaction in the continuum limit described by axial field,

$$(i\partial\!\!\!/ + \gamma^5 A\!\!\!/ - m)\Psi = 0 \qquad \text{Dirac}$$

$$C_0\,(\partial^k{}_j A^j - \Box A^k) - C_2\, A^k = 12\pi^2\, \overline{\Psi}\gamma^5\gamma^k\Psi \qquad \text{Yang/Mills}$$

- all loop corrections of quantum field theory
- axial field is massive (no Higgs mechanism needed)
- C_0 and C_2 determine coupling constant and rest mass,
  depend on regularization,
  computable for given regularization
- non-causal corrections: $-f_{[0]} * j^k_a + 6 f_{[2]} * A^k_a$
  new types of higher order corrections

A more realistic model


AMS/IP
Studies in
Advanced
Mathematics
S.-T. Yau, Series Editor

The Principle of the Fermionic Projector

Felix Finster

American Mathematical Society · International Press

“The Principle of the Fermionic Projector”
AMS/IP Studies in Advanced Math. 35 (2006)

Describe the vacuum by 24 Dirac seas:

$$P(x,y) = \bigoplus_{a=1}^{8} X_a \sum_{\beta=1}^{3} P^{\text{sea}}_{m_{a,\alpha}}(x,y)$$

Results in the continuum limit

The effective interaction can be described by chiral potentials corresponding to the local gauge group

$$SU(2) \times SU(3) \times U(1)^3 \ .$$

- The $SU(3)$ corresponds to an unbroken gauge symmetry. The $SU(3)$ gauge fields couple to the quarks exactly as the strong gauge fields in the standard model.
- The $SU(2)$ potentials are left-handed, coupling to leptons and quarks as in standard model. The $SU(2)$ gauge symmetry is spontaneously broken.
- The $U(1)$ of electrodynamics can be identified with an abelian subgroup of the effective gauge group.

The model has striking similarity to the standard model

- formulation covariant, Ricci tensor comes into play

The local correlation matrices

Back to the analysis (general f, m, n):
Recall that $P(H)$ f-dimensional, negative definite, and $(-P) \geq 0$.
Thus P can be represented as

$$P\psi = -\sum_{\ell=1}^{f} \Psi_\ell \, \langle \Psi_\ell | \psi \rangle$$

again introduce local correlation matrices by

$$(F_x)^i_j = -\langle \Psi_i | E_x \Psi_j \rangle$$

have following properties:

- F_x are Hermitian $f \times f$ matrices of rank $\leq 2n$
- have at most n positive and n negative eigenvalues
- $\sum_{x \in M} \mathrm{Tr}(F_x) = \mathrm{Tr}(P) = f$ gives trace constraint
  (other constraints similarly)

Reformulation of the variational principle

Proposition

- *fermion system can be reconstructed from F_x*
- *variational principle can be expressed in terms of F_x*

Thus formulate variational principle purely in terms of F_x

$$\mathfrak{F} := \{\text{Hermitian } (f \times f)\text{-matrices with above properties}\}$$

consider mappings $\qquad F : M \to \mathfrak{F} : x \mapsto F_x$

minimize $\qquad \mathcal{S} = \sum_{x,y \in M} \mathcal{L}[F(x)F(y)]$

no indefinite inner product spaces needed!
non-compact gauge freedom disappears

Geometric structures

the setup is very simple,
nevertheless geometric structures are encoded

explain in simplest example $n = 1$. Fix $x \in M$, diagonalize F_x,

$$F_x = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & -\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

For products $F(x)F(y)$ consider only upper 2×2 block

$$F_y = \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix}$$

Geometric structures

$$F_x\,F_y \;\simeq\; A_{xy} := \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\beta} \end{pmatrix}$$

A_{xy} is symmetric w.r.to inner product

$$\prec .|. \succ := \left\langle .,\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \right\rangle_{\mathbb{C}^2}$$

can be represented as

$$A_{xy} = \rho\,\mathbb{1} + v_0\,\sigma^3 + i\,v_1\,\sigma^1 + i\,v_2\,\sigma^2$$

The eigenvalues of A_{xy} and the Lagrangian are

$$\lambda_\pm = \rho \pm \sqrt{v_0^1 - v_1^2 - v_2^2}$$

$$\mathcal{L}[F_x\,F_y] = \frac{1}{4}\,(|\lambda_+| - |\lambda-|)^2 = \max(0, v_0^1 - v_1^2 - v_2^2)$$

Geometric structures

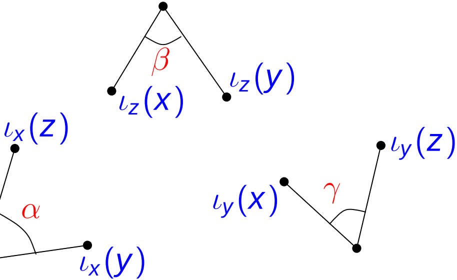
Thus consider $v = (v_0, v_1, v_2)$ as a vector in 3-dimensional Minkowski space $\mathbb{R}^{1,2}$.

$$\mathcal{L}[F_x F_y] = \max(0, \langle v, v \rangle)$$

Gives rise to mapping

$$\iota_x \; : \; M \to \mathbb{R}^{1,2}$$

Measure angles in Minkowski space, for a “triangle” $x, y, z \in M$



Geometric structures

Use Gauß-Bonnet to define “sectional curvature” by

$$\int_{\Delta} K = \alpha + \beta + \gamma - \pi$$

also possible (but more complicated to explain):

- spinor bundle, spin connection, curvature
- parallel transport, geodesics, . . .

has hardly been explored, many open problems!

Generalization to continuum space-times

replace $M = \{1, \ldots, m\}$ by a general measure space (M, μ)

$$\mathbb{M} := \{F : M \to \mathfrak{F} \text{ measurable}\} .$$

$$\text{minimize} \quad \mathcal{S} = \iint_{M \times M} \mathcal{L}[F(x)F(y)]\, d\mu_x\, d\mu_y \quad \text{on } \mathbb{M}$$

- specify $\mathcal{L}$
- impose suitable constraints

gives well-defined variational principle

Theorem (to appear in J. reine angew. Math.)

Minimizers exist in many situations under general assumptions.

Spontaneous structure formation

$$F \;:\; (\text{measure space } M) \longrightarrow (\text{manifold } \mathcal{F})$$

- F induces a topology on M
- F induces a causal structure on M
- as before: induces curvature quantities, . . .

Ingredients of the proof

- Obvious idea: As $\mathcal{S} \geq 0$, choose minimizing sequence F_k, try to construct convergent subsequence

  does not work because of invariance under

  $$F \to F \circ \phi\,, \qquad \phi : M \to M \text{ measure-preserving}$$

- To avoid this problem, introduce measure on $\mathfrak{F}$,

  $$\rho(\Omega) := \mu(F^{-1}(\Omega))$$

  gives sequence ρ_n of measures on $\mathfrak{F}$,

  $$\mathcal{S} = \iint_{\mathfrak{F}\times\mathfrak{F}} \mathcal{L}[pq]\, d\rho_p\, d\rho_q$$

  At the end: reconstruct F from minimizing ρ

Ingredients of the proof

- If $\mathfrak{F}$ were compact, one could apply Banach-Alaoglu and Riesz representation theorem

  $$\rho_n \xrightarrow{C^0(\mathfrak{F})^*} \rho \quad \text{regular Borel measure on } \mathfrak{F}$$

  action continuous under weak-*-convergence

  but $\mathfrak{F}$ is in general non-compact manifold,
  the support of the measures ρ_k may “escape to infinity”
- Use homogeneity of the functionals

  $$\mathrm{Tr}(\nu A) = \nu\, \mathrm{Tr}(A)\,, \qquad \mathcal{L}[\nu A] = |\nu|^2\, \mathcal{L}[A], \ldots$$

Ingredients of the proof

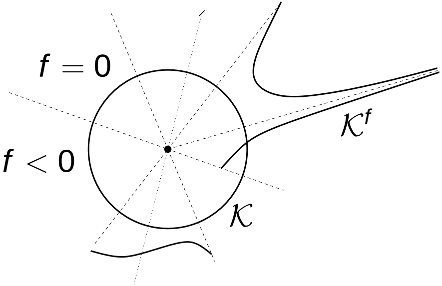
$$\mathcal{K} = \{p \in \mathfrak{F} \text{ with } \|p\| = 1\} \cup \{0\} \quad \text{compact manifold}$$

Introduce measures on $\mathcal{K}$, so-called moment measures

$$\mathfrak{m}^{(0)}(\Omega) = \frac{1}{2}\,\rho(\mathbb{R}\Omega \setminus \{0\}) + \rho(\Omega \cap \{0\})$$
$$\mathfrak{m}^{(1)}(\Omega) = \frac{1}{2}\int_{\mathbb{R}^+\Omega} \|p\|\, d\rho(p) \;-\; \frac{1}{2}\int_{\mathbb{R}^-\Omega} \|p\|\, d\rho(p)$$
$$\mathfrak{m}^{(2)}(\Omega) = \frac{1}{2}\int_{\mathbb{R}\Omega} \|p\|^2\, d\rho(p)$$

express variational principle in terms of moment measures, realize moment measures by measure on a graph over $\mathcal{K}$:

$$\mathcal{K}^f = \{f(p)\,p \text{ with } p \in \mathcal{K}\} \subset \mathfrak{F}$$



Work with sequence of measurable functions f_k converges pointwise, apply Fatou's lemma

$$\int_{\mathcal{K}} \lim_{k\to\infty} |f_k|^2 d\mathfrak{m}^{(0)} \leq \lim_{k\to\infty} \int_{\mathcal{K}} |f_k|^2 d\mathfrak{m}^{(0)}$$

bubbling phenomena might occur, need to be ruled out

Open problem

known: $f \in L^4(\mathcal{K}, d\mathfrak{m}^{(0)})$, but $d\mathfrak{m}^{(0)}$ might be singular w.r.to $d\mu_{\mathcal{K}}$

- Is $\mathcal{K}^f$ a submanifold? Of what co-dimension? Is $\mathcal{K}^f$ bounded?
- What is the regularity? Is $\mathcal{K}_f$ smooth?
- Can bubbling really occur?
- Are minimizers unique?
  At least up to isomorphism of M and $\mathfrak{F}$?

A counter example to compactness

so-called three-dimensional Dirac sphere

Choose $f = 4$, $n = 2$ and set $M = S^3$.

Introduce the four 4×4-matrices

$$\gamma^\alpha = \begin{pmatrix} \sigma^\alpha & 0 \\ 0 & -\sigma^\alpha \end{pmatrix}, \quad \alpha = 1, 2, 3 \qquad \text{and} \qquad \gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

(are generators of the Euclidean Clifford algebra)

For any $\tau > 1$ set

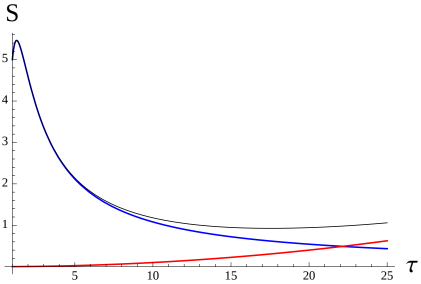
$$F \,:\, M \to \mathcal{F} \,:\, x \mapsto \sum_{i=1}^{4} \tau\, x^i \gamma^i + \mathbb{1}$$

The trace constraint is satisfied.

$$\mathcal{S}[F] = \frac{512}{15\pi}\,\frac{1}{\tau} + \mathcal{O}(\tau^{-2})$$

divergent minimizing sequence!

A counter example to compactness



for normalized counting measure, $m = \#M$

$$\mathcal{S} = \frac{1}{m^2} \sum_{x,y \in M} \mathcal{L}[F(x)\,F(y)] \geq \frac{1}{m^2} \sum_{x \in M} \mathcal{L}[F(x)\,F(x)] = 16\,\frac{\tau^2}{m}$$

thus discreteness gives rise to additional red contribution

- allows to construct discrete systems with very small action
- are indeed closely related to Dirac sea configurations