Immersions of pseudo-Riemannian manifolds into indefinite Euclidean spaces

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Presentation

1. Introduction
   - Classical Weierstraß representations
   - Para-complex geometry
2. Results
   1. A Weierstraß representation for pseudo-Riemannian manifolds.
   2. Associated families
   3. Examples

Theorem (Weierstraß 1866)

Let M be a Riemannian surface. The two following statements are equivalent :

1) *The map $F = (F_1, F_2, F_3) : M \to \mathbb{R}^3$ is a minimal conformal immersion.*
2) *There exists a triple $\omega = (\omega_1, \omega_2, \omega_3)$ of holomophic $(1,0)$-forms on M such that*
   (i) $\sum_i \omega_i^2 = 0$,
   (ii) *The forms $\Re\omega_i$ are exact.*

   satisfying

$$F(q) = \Re \int_p^q (\omega_1, \omega_2, \omega_3) + Constant.$$

- The F_i are harmonic : $0 = \Delta F_i$
- $F = \Re(f)$, $f : M \to \mathbb{C}^3$ (holomorphic representative),
- $\omega_i = \phi_i dz$ with $\phi := \frac{\partial f}{\partial z} = (\phi_1, \phi_2, \phi_3)$ in

$$Q = \{(z_1^2, z_2^2, z_3^2) \in \mathbb{C}^3 | z_1^2 + z_2^2 + z_3^2 = 0\},$$

- There exists an associated family $\{F_\theta = \theta \in \mathbb{R}\}$, $F_0 = F$ given by

$$F_\theta = \Re(e^{i\theta f})$$

of "isometric deformations".

Other results.

- **Kenmotsu** : Representation for CMC-surfaces in R^3.
- **Konopelchenko, Kenmotsu, Bobenko** : Representations of minimal and CMC surfaces in R^n,
- **Abresch, Kusner-Schmitt '99, Taimanov...** : spinorial Weierstraß representations,
- **Dajzer-Gromoll '85-'95, Eschenburg-Tribuzy-Ferreira,...** : Kähler submanifolds of spaceforms,
  Arezzo-Pirola-Solci '01 : Weierstraß representation of pluriminimal Kähler manifolds in Euclidean space.

The algebra C of para-complex numbers is the real algebra generated by 1 and the para-complex unit e, such that $e^2 = 1$. For all $z = x + ey$ one define

1) The para-complex conjugation :

$$\begin{aligned} \bar{\cdot} : C &\to C \\ x + ey &\mapsto x - ey \end{aligned}$$

2) real and imaginary parts

$$\Re(z) = \frac{z + \bar{z}}{2} = x, \quad \Im(z) = \frac{(z - \bar{z})e}{2} = y$$

Remark : $C \cong \mathbb{R} \oplus \mathbb{R} \cong \mathcal{C}l_{0,1}$.

Definition

Let V a finite dimensional real vector space. A ***para-complex structure*** *is an endomorphism $J \in End(V)$, such that*

$$J^2 = Id, \quad dimV^+ = dimV^-,$$

with $V^\pm = Ker(Id \mp J)$.
We call (V, J) a ***para-complex vector space.***

Definition

An ***almost para-complex structure*** *on a manifold M is an endomorphism field $J \in \Gamma(End(TM))$ such that, for all $p \in M$, J_p is a para-complex structure on T_pM.*
J is ***integrable*** *iff the distributions $T^\pm M := ker(Id \mp J)$ are integrable.*
An ***para-complex structure*** *on M is an integrable almost para-complex structure on M. We call (M, J)* ***para-complex manifold.***
$dim_C M = \frac{dimM}{2}$.

Let (M, J) be a para-complex manifold. $TM = TM^+ \oplus TM^-$.

1. "adapted" coordinates : Frobenius
   $\Rightarrow$ There exists an open neighborhood U of M, and functions $z^i_\pm$, $i = 1, \ldots, n$ on U
   - which are constant on the leaves of $TM^\mp$,
   - whose differential $dz^i_\pm$ are linearly independent.

   $\Rightarrow (z^1_+, \ldots, z^n_+, z^1_-, \ldots, z^n_-)$ is a local coordinates system.
2. Let
   $$x_i = \frac{z^i_+ + z^i_-}{2}, \quad y_i = \frac{z^i_+ - z^i_-}{2}.$$
   It is a system of real local coordinates on U.

3. Para-holomorphic coordinates :

Definition

A smooth map $\phi : (M, J_M) \rightarrow (N, J_N)$ *is para-holomorphic iff* $d\phi J_M = J_N d\phi$.

A para-holomorphic local coordinates system is a system of para-holomorphic functions z^i, $i = 1, \ldots, n$ on $U \subset M$, such that $x_i = \Re(z^i)$, $y_i = \Im(z^i)$.

- $(p+, q-)$-forms : The decomposition $TM = TM^+ \oplus TM^-$ extends to

$$\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^* M.$$

and to the differential forms on M :

$$\Omega^k(M) = \bigoplus_{k=p+q} \Omega^{p+,q-}(M).$$

$$d : \Omega^k(M) \to \Omega^{k+1},$$

$$\partial^+ : \Omega^{p+,q-}(M) \to \Omega^{(p+1)+,q-}, \quad \partial^- : \Omega^{p+,q-}(M) \to \Omega^{p+,(q+1)-}.$$

- (p, q)-forms : $TM^C := TM \otimes C = TM^{1,0} \oplus TM^{0,1}$, avec

$$T_pM^{1,0} := \{X + eJX | X \in T_pM\}, \; T_pM^{0,1} := \{X - eJX | X \in T_pM\},$$

$$\Rightarrow \Lambda^k T^* M^C = \bigoplus_{k=p+q} \Lambda^{p,q} T^* M, \quad \Omega^k_C(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M).$$

Consider $\mathbb{R}^{p,q}$ with standard indefinite metric :

$$dx_1^2 + \ldots + dx_p^2 - dx_{p+1}^2 - \ldots - dx_{p+q}^2.$$

Let $M^{m,m}$ be para-complex, $f : M^{m,m} \to \mathbb{R}^{p,q}$ a smooth map.

$$f(q) = \int_p^q df + f(p) = \Re \int_p^q (\omega_1, \ldots \omega_{p+q}) + f(p),$$

where $\{\omega_i\}_{i=1,\ldots,p+q}$ are $(1,0)$-forms, $\Re\omega_i$ exact.

$$\omega_i(z_1, \ldots z_m) = \sum_j^m \Omega_{ij}(z_1, \ldots z_m) dz_j,\ \Omega_{ij}(z_1, \ldots z_m) = \left(\frac{\partial f_i}{\partial z_j}\right)_{ij}.$$

Define the maps

$$\begin{array}{rcl} \phi_j : U & \mapsto & C^n \\ (z_1, \ldots z_m) & \to & \dfrac{\partial f}{\partial z_j} = \dfrac{\partial f}{\partial x_j} + e\dfrac{\partial f}{\partial y_j}, j = 1, \ldots m. \end{array}$$

Theorem (L.)

Let $M^{m,m}$ be a para-complex manifold. Then the following two conditions are equivalent :

1. *The map $f : M^{m,m} \rightarrow \mathbb{R}^{p,q}$, $p+q=n$, is a pluri-conformal immersion.*
2. *There exist maps $\phi_j : U \rightarrow C^n$, $j = 1, ..., m$, satisfying the conditions*
   a) *The vector $\phi_1(p), \dots \phi_m(p)$ are linearly independent at every $p \in U$.*
   b) $\phi_i \cdot \underline{\phi_j} = 0, \quad i,j = 1 \dots m,$
   c) $\phi_i \cdot \overline{\phi_j} \neq 0, \quad i,j = 1 \dots m,$
   where " $\cdot$ " is the product $v \cdot w := \sum_{i=1}^{p} v_i w_i - \sum_{i=p}^{p+q} v_i w_i$.
   d) $\frac{\partial \phi_j}{\partial z_k} = \frac{\partial \phi_k}{\partial z_j}, \quad \frac{\partial \phi_j}{\partial \overline{z_k}} = \frac{\partial \overline{\phi_k}}{\partial z_j} \quad i,j = 1 \dots n.$

Proposition

A pluri-conformal immersion is pluriminimal if and only if the maps ϕ_i, $i = 1, \ldots n$ *are para-holomorphic.*

Proposition

Let $M^{m,m}$ *be a para-complex manifold and* $f : M^{m,m} \longrightarrow \mathbb{R}^{p,q}$, $p + q = n$ *be a pluri-conformal immersion. Then the pseudo-Riemannian metric induced on* M *is para-Kähler if and only if the vectors* $\frac{\partial}{\partial \bar{z}_j}\phi_k$ *are normal. Consequently if* f *is pluri-minimal the induced metric is para-Kähler.*

Definition

Let $f : M^{m,m} \to \mathbb{R}^{p,q}$ be a para-Kählerian immersion, α its second fundamental form.
For $\theta \in \mathbb{R}$, define :

$$R_\theta : TM \to TM, \quad R_\theta = e^{J\theta} = cosh(\theta)I + sinh(\theta)J,$$

An associated family of f is a one-parameter family of para-Kählerian immersions $f_\theta : M^{m,m} \to \mathbb{R}^{p,q}$, $f_0 = f$, with second fundamental form α_{f_θ} such that

$$\psi_\theta(\alpha_{f_\theta}(X,\ Y)) = \alpha_\theta(X,\ Y) := \alpha(R_\theta X,\ R_\theta Y),$$

for some bundle isomorphisms $\psi_\theta : f_\theta^* TN \to f_\theta^* TN$.

Theorem (L.)

Let $f : M^{m,m} \longrightarrow \mathbb{R}^{p,q}$ be a para-Kählerian immersion. Then there exists an associated family of f if and only if f has parallel pluri-mean curvature and $R^N(f^* T^{1,0}M, f^* T^{1,0}M) = 0$.

Remark : Pluri-minimal immersions have an associated family.

- Case m=1, M Lorentzian surface in $\mathbb{R}^{2,1}$:
  - Konderak '01 : Para-complex Weierstraß representation for minimal Lorentz surfaces,
  - L. '06 : Real and para-complex Weierstrass representation for general Lorentz surfaces
    $\Rightarrow$ spinorial representation,
  - Analogs of Enneper surface, Catenoid and helicoid (associated families).
- Case m arbitrary : Work in progress !