The Einstein-Maxwell equations and the complex hyperbolic plane

Alan D. Rendall,
Max Planck Institute for Gravitational Physics,
Albert Einstein Institute,
Am Mühlenberg 1,
14476 Golm, Germany.

Introduction

The fundamental object in general relativity is a Lorentzian metric g on a four-dimensional manifold M

In general there are other objects on M describing matter fields

An electromagnetic field is described by a two-form F

The Einstein equations are given by $\mathrm{Ric}(g) - \frac{1}{2}\mathrm{Scal}(g)g = 8\pi T$

Here T is the energy-momentum tensor

In the absence of matter $T = 0$ (vacuum case)

For an electromagnetic field $T_{\alpha\beta} = \frac{1}{4\pi}(F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \frac{1}{4}F_{\gamma\delta}F^{\gamma\delta}g_{\alpha\beta})$

F satisfies the equations $dF = 0$ and $d^*F = 0$

Coupled to the Einstein equations this gives the Einstein-Maxwell system

A central task in mathematical relativity is understanding global properties of general solutions of the Einstein-matter equations

This is very difficult (even for vacuum)

Solutions with symmetry

Simplifications result from symmetry assumptions

Ex: T^2 symmetry

Torus T^2 acts on M with spacelike orbits diffeomorphic to T^3

g and F are invariant under the action

Let G_1 be the group of isometries of the plane generated by translations and $(x, y) \mapsto (-x, -y)$

Assume that the action of T^2 extends to an action of G_1 by symmetries

This is known as Gowdy symmetry

Cf. E. Nungesser and ADR (2009)

It is a major simplification

In vacuum case a system of two semilinear wave equations in one space dimension results

With electromagnetic field there are four equations

The Gowdy equations

Assuming Gowdy symmetry the metric can be written as

$$t^{-1/2}e^{\lambda/2}(-dt^2+d\theta^2)+t[e^P(dx+Q\,dy)^2+e^{-P}dy^2]$$

P, Q and λ are real-valued functions of (t,θ)

$(t,\theta)\in(0,\infty)\times S^1$

The main equations are

$$\begin{aligned} P_{tt}+t^{-1}P_t &= P_{\theta\theta}+e^{2P}(Q_t^2-Q_\theta^2) \\ Q_{tt}+t^{-1}Q_t &= Q_{\theta\theta}-2(P_tQ_t-P_\theta Q_\theta) \end{aligned}$$

There exists a unique global solution for smooth data on $t = t_0$

Asymptotics as $t \to 0$, $t \to \infty$?

Let G_2 be generated by translations, $x \mapsto -x$, $y \mapsto -y$

Polarized case: action of G_1 extends to action of G_2

This is equivalent to $Q = 0$

Within Gowdy symmetry equations for polarized Einstein-Maxwell equivalent to general Einstein vacuum

Wave maps

Let (M, h_1) be a pseudo-Riemannian manifold and (N, h_2) a Riemannian manifold

If $\Phi : M \to N$ is a smooth map let $L = |T\Phi|^2$

Consider the corresponding Euler-Lagrange equations

These define a harmonic map if h_1 is Riemannian and a wave map if h_1 is Lorentzian.

Let g_0 be the following three-dimensional Lorentzian metric

$$-dt^2 + d\theta^2 + t^2 d\phi^2$$

Then the Gowdy equations are equivalent to those for a wave map from g_0 to the hyperbolic plane which is independent of ϕ

Metric of the hyperbolic plane is written as $dP^2 + e^{2P} dQ^2$

The Einstein-Maxwell equations with Gowdy symmetry can be related to the complex hyperbolic plane

Discovered in the context of black hole uniqueness theorems

In that case an Abelian group acts on timelike hypersurfaces

Electromagnetic Gowdy story goes back to work of Mansfield, Moncrief

It is assumed that F comes from a potential $F = dA$

Due to symmetry only $\omega = A_2$ and $\chi = A_3$ are non-vanishing

Non-local transformation to get to wave map form

$$Q_\theta = -t^{-1} e^{-2P} (\psi_t + 2\omega\eta_t), \qquad Q_t = -t^{-1} e^{-2P} (\psi_\theta + 2\omega\eta_\theta),$$
$$2(\chi_t - Q\omega_t) = e^{-P} \eta_\theta, \qquad 2(\chi_\theta - Q\omega_\theta) = e^{-P} \eta_t.$$

It is also convenient to define $\gamma = \frac{1}{2}(P + \log t)$ and $\tilde{\omega} = 2\omega$

The main Einstein-Maxwell equations are then

$$\gamma_{tt} + t^{-1}\gamma_t = \gamma_{\theta\theta} - \frac{1}{4}e^{-2\gamma}(\tilde{\omega}_t^2 - \tilde{\omega}_\theta^2 + \eta_t^2 - \eta_\theta^2)$$
$$-\frac{1}{2}e^{-4\gamma}[(\psi_t + \tilde{\omega}\eta_t)^2 - (\psi_\theta + \tilde{\omega}\eta_\theta)^2]$$
$$\psi_{tt} + t^{-1}\psi_t = \psi_{\theta\theta} + 2\tilde{\omega}(\gamma_t\eta_t - \gamma_\theta\eta_\theta) + 4(\gamma_t\psi_t - \gamma_\theta\psi_\theta)$$
$$-(1 - \tilde{\omega}^2 e^{2\gamma})(\tilde{\omega}_t\eta_t - \tilde{\omega}_\theta\eta_\theta) + \tilde{\omega}e^{2\gamma}(\psi_t\tilde{\omega}_t - \psi_\theta\tilde{\omega}_\theta)$$
$$\tilde{\omega}_{tt} + t^{-1}\tilde{\omega}_t = \tilde{\omega}_{\theta\theta} + 2(\gamma_t\tilde{\omega}_t - \gamma_\theta\tilde{\omega}_\theta)$$
$$+e^{-2\gamma}[\eta_t(\psi_t + \tilde{\omega}\eta_t) - \eta_\theta(\psi_\theta + \tilde{\omega}\eta_\theta)]$$
$$\eta_{tt} + t^{-1}\eta_t = \eta_{\theta\theta} + 2(\gamma_t\eta_t - \gamma_\theta\eta_\theta)$$
$$-e^{-2\gamma}[(\psi_t\tilde{\omega}_t - \psi_\theta\tilde{\omega}_\theta) + \tilde{\omega}(\eta_t\tilde{\omega}_t - \eta_\theta\tilde{\omega}_\theta)]$$

These equations are the equations for a wave map from g_0 to the complex hyperbolic plane

Metric of the target space is written as

$$4d\gamma^2 + e^{-2\gamma}(d\tilde{\omega}^2 + d\eta^2) + e^{-4\gamma}(d\psi + \tilde{\omega}d\eta)^2$$

Vacuum and polarized cases arise by restricting range to totally geodesic submanifolds, real and holomorphic respectively

Asymptotics for $t \to \infty$ (vacuum case)

Ringström, H. 2004 On a wave map equation arising in general relativity. *Commun. Pure Appl. Math.* **57**, 657–703

$\mathcal{E} = \int_{S^1} [P_t^2 + P_\theta^2 + e^{2P}(Q_t^2 + Q_\theta^2)]d\theta$, $\frac{d\mathcal{E}}{dt} \leq 0$

Proof has two main steps:

1. Show that $\lim_{t\to\infty} \mathcal{E}(t) = 0$

2. Obtain detailed asymptotics

P and Q define a loop in the hyperbolic plane for fixed t

As $t \to \infty$ the diameter of this loop decreases like $t^{-\frac{1}{2}}$

The 'centre' of the converges to a curve in the hyperbolic plane

This is a circle in the half-plane model (constant geodesic curvature)

Motion along the curve can be described precisely

Isometry group of the hyperbolic plane leads to Noether identities

$\nabla_\alpha \left(X^A \frac{\partial L}{\partial(\nabla_\alpha \phi^A)} \right) = 0$, X^A any Killing vector

This shows that the following quantities are conserved

$$A = \int_{S^1} [2Q(tQ_t)e^{2P} - 2(tP_t)]d\theta$$
$$B = \int_{S^1} [e^{2P}(tQ_t)]d\theta$$
$$C = \int_{S^1} [(tQ_t)(1 - e^{2P}Q^2) + 2Q(tP_t))]d\theta$$

The Casimir invariant $A^2 + 4BC$ determines the type of circle

Essential use of conserved quantities in determining asympotics

Asymptotics for $t \to \infty$ (Maxwell case)

There exists an analogue of $\mathcal{E}$

It tends to zero as $t \to \infty$ (Ringström 2006)

What does the centre of the loop do asymptotically?

Eight conserved quantities are available. Two vanish identically for solutions of the Einstein-Maxwell equations

Conclusions and outlook

1. For the Gowdy solutions of the Einstein vacuum equations the late-time asymptotics are well understood

2. The geometric interpretation of the Gowdy equations as a wave map with values in the hyperbolic plane played a central role in this proof

3. For solutions of the Einstein-Maxwell equations with Gowdy symmetry a wave map interpretation is also available, with the target space being the complex hyperbolic plane

4. In this case the late-time asymptotics are only partly understood. Are there essentially new phenomena?

5. The wave map interpretation may be the key to completing the picture