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Thema: **Real Z_2-bi-graded Hermitian Clifford modules and the STM action.**

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Real $\mathbb{Z}_2-$bi-graded Hermitian Clifford modules and the Standard Model action

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Content

1 Brief motivation

The action functional of the **Standard Model**:

$$\mathcal{I}_{\mathrm{STM}} = \int_M \mathcal{L}_{\mathrm{STM}}\, dvol_{\mathrm{M}}\,, \tag{1}$$

whereby

$$\mathcal{L}_{\mathrm{STM}} =$$

$$-\partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2\, W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2} M^2\, Z^0_\mu Z^0_\mu$$

$$-\frac{1}{2}\partial_\nu A_\mu \partial_\nu A_\mu - \frac{1}{2}\partial_\nu H \partial_\nu H - \frac{1}{2} m_h^2 H^2 - \beta_h [\frac{2M^2}{g^2} + \frac{2M}{g}\, H + \frac{1}{2} H^2]$$

$$-igc_w[\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu)$$

$$+Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] + \cdots + \text{more than 50 additional terms}$$

This Standard Model Lagrangian can be described in terms of a **specific class of Dirac operators**:

$$\not{P}_{\mathrm{D}} \quad = \quad \begin{pmatrix} \not{D}_{\mathcal{E}} & -\not{F}_{\mathrm{D}} \\ \not{F}_{\mathrm{D}} & \not{D}_{\mathcal{E}} \end{pmatrix}, \tag{2}$$

defined on the doubling

$$({}^{2}\mathcal{E} \equiv \mathcal{E} \oplus \mathcal{E}, \tau_{{}^{2}\mathcal{E}} \equiv \tau_{\mathcal{E}} \ominus \tau_{\mathcal{E}}, \gamma_{{}^{2}\mathcal{E}} \equiv \gamma_{\mathcal{E}} \oplus \gamma_{\mathcal{E}}) \longrightarrow (M, g_{\mathrm{M}}) \tag{3}$$

of a $\mathbb{Z}_2-$graded Clifford module

$$(\mathcal{E} = \mathcal{E}^{+} \oplus \mathcal{E}^{-}, \gamma_{\mathcal{E}}) \longrightarrow (M, g_{\mathrm{M}})\,. \tag{4}$$

Here,

$$\not{D}_{\mathcal{E}} \equiv i\not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} \tag{5}$$

belongs to the distinguished class of **simple type Dirac operators**.

In fact, when evaluated with respect to (2), the **Dirac action**:

$$\mathcal{I}_{\rm D,\, tot}: \quad \mathcal{D}({}^2\mathcal{E}) \times \mathfrak{S}ec(M,\, {}^2\mathcal{E}) \longrightarrow \mathbb{C}$$
$$(\not{\mathcal{D}}, \Psi) \mapsto \int_M *\big(\langle\Psi, \not{\mathcal{D}}\Psi\rangle_{{}^2\mathcal{E}} + \mathrm{tr}_\gamma\big(curv(\not{\mathcal{D}}) - \mathrm{ev}_g(\omega_{\rm D}^2)\big)\big), \quad (6)$$

decomposes into the various parts of the Standard Model action, including gravity described in terms of the Einstein-Hilbert functional. In particular, the **fermionic part of the total Dirac action** reduces to the fermionic part of the STM action:

$$\int_M \langle\Psi, \not{P}_{\rm D}\Psi\rangle_{{}^2\mathcal{E}}\, dvol_{\rm M} = \int_M \langle\psi, (i\not{\partial}_{\rm A} + \phi_{\mathcal{E}})\psi\rangle_{\mathcal{E}}\, dvol_{\rm M}\,, \quad (7)$$

provided the sections $\Psi \in \mathfrak{S}ec(M, {}^2\mathcal{E})$ on the doubled Clifford module are restricted to **diagonal sections** $\Psi := {}^2\psi \equiv (\psi, \psi)$ and the sections $\psi \in \mathfrak{S}ec(M, \mathcal{E})$ are restricted, furthermore, to the **physical sub-module** $\mathcal{E}_{\rm phys} \hookrightarrow \mathcal{E} \longrightarrow M$ of the underlying Clifford module.

Basic Question:

What is the geometric structure, which underlies Pauli type Dirac operators:

$$({}^2\mathcal{E},\, {}^2\psi,\, \not{P}_{\mathrm{D}})\,? \tag{8}$$

An answer is provided by a careful analysis of the algebraic and geometric structure of the **Dirac equation** and the **Majorana equation**:

$$i\partial\!\!\!/\chi = m_{\mathrm{D}}\chi \quad \Leftrightarrow \quad \begin{cases} i\partial\!\!\!/\chi_{\mathrm{R}} = m_{\mathrm{D}}\chi_{\mathrm{L}}\,, \\ i\partial\!\!\!/\chi_{\mathrm{L}} = m_{\mathrm{D}}\chi_{\mathrm{R}}\,, \end{cases} \tag{9}$$

$$i\partial\!\!\!/\chi = m_{\mathrm{M}}\chi^{\mathrm{cc}} \quad \Leftrightarrow \quad \begin{cases} i\partial\!\!\!/\chi_{\mathrm{R}} = m_{\mathrm{M}}\chi_{\mathrm{R}}^{\mathrm{cc}}\,, \\ i\partial\!\!\!/\chi_{\mathrm{L}} = m_{\mathrm{M}}\chi_{\mathrm{L}}^{\mathrm{cc}}\,, \end{cases} \tag{10}$$

where, respectively, χ_{R}, χ_{L} are the "chiral" eigen states: $\chi = \chi_{\mathrm{R}} + \chi_{\mathrm{L}}$, m_{D} is the "Dirac mass", m_{M} the "Majorana mass" and "*cc*" has the physical meaning of "charge conjugation".

2 Real Clifford modules and the Pauli map

2.1 Some general facts:

Let $(\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-, \gamma_{\mathcal{E}}) \longrightarrow (M, g_{\rm M})$ be a **general Clifford module** with an **odd Clifford mapping**:

$$\gamma_{\mathcal{E}}(\alpha)^2 = \varepsilon\, g_{\rm M}(\alpha,\alpha)\, {\rm id}_{\mathcal{E}}\,, \tag{11}$$

for all $\alpha \in T^*M$. It follows that

- Canonical one-form: $\Theta \overset{\rm loc.}{=} \frac{\varepsilon}{n}\, e^k \otimes \gamma_{\mathcal{E}}(e_k^\flat) \in \Omega^1(M, {\rm End}^-(\mathcal{E}))\,;$

$$\Rightarrow \qquad ext_\Theta : \Omega^0(M, {\rm End}^\pm(\mathcal{E})) \longrightarrow \Omega^1(M, {\rm End}^\mp(\mathcal{E})) \\ \Phi \mapsto \Theta \wedge \Phi\,, \tag{12}$$

right-inverse of the "quantization map":

$$\begin{array}{rcl} \delta_\gamma : \Omega^*(M, \mathrm{End}(\mathcal{E})) & \longrightarrow & \mathfrak{S}ec(M, \mathrm{End}(\mathcal{E})) \\ \omega \otimes \chi & \mapsto & \gamma_{\mathcal{E}}(\sigma_{\mathrm{Ch}}^{-1}(\omega)) \circ \chi \,. \end{array} \tag{13}$$

- **Clifford connections:**

$$\mathcal{A}_{\mathrm{Cl}}(\mathcal{E}) := \{\partial_{\mathrm{A}} \in \mathcal{A}(\mathcal{E}) \,|\, \partial_{\mathrm{A}}\Theta \equiv 0\} \,. \tag{14}$$

- First and second order decomposition:

$$\not{D} = \not{\partial}_{\mathrm{B}} + \Phi_{\mathrm{D}} \,, \tag{15}$$

$$\not{D}^2 = \triangle_{\mathrm{B}} + V_{\mathrm{D}} \,. \tag{16}$$

Here, respectively, $\not{\partial}_{\mathrm{B}} \equiv \delta_\gamma \circ \partial_{\mathrm{B}}$ is the quantization of the **Bochner connection** $\partial_{\mathrm{B}} \in \mathcal{A}(\mathcal{E})$ that is uniquely defined by $\not{D}$

via

$$2\, ev_g(df, \partial_{\rm B}\psi) := \varepsilon \Big([\not{D}^2, f] - \delta_{\rm g} df \Big) \psi \,, \qquad (17)$$

for all $f \in \mathcal{C}^\infty(M)$ and $\psi \in \mathfrak{S}ec(M, \mathcal{E})$; $\triangle_{\rm B} := \varepsilon ev_g(\partial_{\rm B} \circ \partial_{\rm B})$ is the corresponding **Bochner-Laplacian** (or "trace/connection Laplacian").

- **Dirac connections**:

$$\partial_{\rm D} \quad := \quad \partial_{\rm B} + \omega_{\rm D} \,, \qquad (18)$$

$$\omega_{\rm D} \quad \equiv \quad ext_\Theta(\Phi_{\rm D}) \qquad \textbf{Dirac form} \,, \qquad (19)$$

whereby $\not{\partial}_{\rm D} \equiv \delta_\gamma \circ \partial_{\rm D} = \not{D}$.

- **The Dirac-Lagrangian:**

$$\begin{aligned} *\mathcal{L}_{\rm D} \quad &:= \quad {\rm tr}_{\mathcal{E}} V_{\rm D} \\ &= \quad {\rm tr}_\gamma \big(curv(\not{D}) - \varepsilon ev_g(\omega_{\rm D}^2) \big) + div \xi_{\rm D} \,. \end{aligned} \qquad (20)$$

Here, $\xi_{\mathrm{D}} := -\varepsilon\, \mathrm{tr}_{\mathcal{E}} \omega_{\mathrm{D}}^{\sharp} \in \mathfrak{S}ec(M, TM)$ is the **Dirac vector field** and

$$curv(\not{D}) := \partial_{\mathrm{D}} \wedge \partial_{\mathrm{D}} \in \Omega^2(M, \mathrm{End}(\mathcal{E})) \tag{21}$$

is the **curvature of the Dirac operator** $\not{D} \in \mathcal{D}(\mathcal{E})$ and $\mathrm{tr}_\gamma \equiv \mathrm{tr}_{\mathcal{E}} \circ \delta_\gamma$ is the "quantized trace".

- **Dirac operators of simple type:**

$$\not{D} = \not{\partial}_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathrm{D}}\,, \tag{22}$$

where $\phi_{\mathrm{D}} \in \mathrm{Sec}(M, \mathrm{End}^-_\gamma(\mathcal{E}))$.

- **Yang-Mills-Higgs connections:**

$$\not{\partial}_{\mathrm{YMH}} := \not{\partial}_{\mathrm{A}} + \Phi_{\mathrm{H}}\,, \tag{23}$$

where $\Phi_{\mathrm{H}} \in \mathrm{Sec}(M, \mathrm{End}_\gamma(\mathcal{E}))$.

They have the property that the corresponding Dirac connections read:

$$\begin{aligned} \partial_{\rm D} &= \partial_{\rm B} + \Theta \wedge \Phi_{\rm D} \\ &= \partial_{\rm A} + \Phi_{\rm H}\Theta \\ &\equiv \partial_{\rm YMH}\,. \end{aligned} \tag{24}$$

Higgs gauge potential: $H := \Phi_{\rm H}\Theta \in \Omega^1(M, {\rm End}^-(\mathcal{E}))$.

2.2 Real Clifford modules:

2.2.1 Majorana modules:

Definition 2.1 *A Hermitian Clifford module is called a "real $\mathbb{Z}_2-$bi-graded Hermitian Clifford module" ("real Clifford module" for short), if it is endowed, in addition, with a $\mathbb{C}-$linear involution $\tau_{\mathcal{E}}$, making $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \longrightarrow M$ $\mathbb{Z}_2-$graded, and a $\mathbb{C}-$anti-linear* *involution* $J_\mathcal{E}$*, making* $\mathcal{E} = \mathcal{M}_\mathcal{E} \otimes \mathbb{C} \longrightarrow M$ *real, such that*

$$\begin{aligned} \tau_\mathcal{E} \circ \gamma_\mathcal{E}(\alpha) &= -\gamma_\mathcal{E}(\alpha) \circ \tau_\mathcal{E} \,, \\ J_\mathcal{E} \circ \gamma_\mathcal{E}(\alpha) &= \pm\gamma_\mathcal{E}(\alpha) \circ J_\mathcal{E} \,, \\ J_\mathcal{E} \circ \tau_\mathcal{E} &= \pm\tau_\mathcal{E} \circ J_\mathcal{E} \,, \\ \langle J_\mathcal{E}(z), J_\mathcal{E}(w)\rangle_\mathcal{E} &= \pm\langle w, z\rangle_\mathcal{E} \,, \end{aligned} \tag{25}$$

for all $\alpha \in T^*M$ *and* $z, w \in \mathcal{E}$*.*

In particular, a real Clifford module is called a **Majorana module***, provided that*

$$J_\mathcal{E} \circ \tau_\mathcal{E} = -\tau_\mathcal{E} \circ J_\mathcal{E} \,. \tag{26}$$

2.2.2 Dirac modules:

Definition 2.2 *A real Clifford module*

$$(\mathcal{S}, \langle \cdot, \cdot \rangle_{\mathcal{S}}, \tau_{\mathcal{S}}, \gamma_{\mathcal{S}}, J_{\mathcal{S}}) \tag{27}$$

is called a **Dirac module***, provided there is a Majorana module* $(\mathcal{W}, \langle \cdot, \cdot \rangle_{\mathcal{W}}, \tau_{\mathcal{W}}, \gamma_{\mathcal{W}}, J_{\mathcal{W}})$ *over* (M, g_{M})*, such that*

$$\mathcal{S} = {}^{2}\mathcal{W} = \mathcal{W} \otimes \mathbb{C}^{2}\,, \tag{28}$$

$$\tau_{\mathcal{S}} = \mathrm{id}_{\mathcal{W}} \otimes \tau_{2}\,, \tag{29}$$

$$\gamma_{\mathcal{S}} = \gamma_{\mathcal{W}} \otimes \varepsilon_{2}\,, \tag{30}$$

$$J_{\mathcal{S}} = J_{\mathcal{W}} \otimes \varepsilon_{2}\,, \tag{31}$$

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{S}} = \langle u_1, v_2 \rangle_{\mathcal{W}} \pm \langle v_1, u_2 \rangle_{\mathcal{W}}\,, \tag{32}$$

depending on whether $\langle J_{\mathcal{W}}(u), J_{\mathcal{W}}(v)\rangle_{\mathcal{W}} = \pm\langle v, u\rangle_{\mathcal{W}}$*, for all* $u, v \in \mathcal{W}$*.*

Here, $\mathbf{1}_2 \in \mathbb{C}(2)$ and τ_2, ε_2, $\mathrm{I}_2 \in \mathbb{C}(2)$ denote, respectively, the two-by-two unit matrix and

$$\tau_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{I}_2 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{33}$$

2.3 The Pauli map:

Let

$$(\mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{P}}, \tau_{\mathcal{P}}, \gamma_{\mathcal{P}}, J_{\mathcal{P}}) \tag{34}$$

be the doubling of the real Clifford module $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}})$. That is,

$$\mathcal{P} \;:=\; {}^2\mathcal{E} \;\equiv \mathcal{E} \otimes \mathbb{C}^2\,, \tag{35}$$

$$\begin{aligned}
\langle\cdot,\cdot\rangle_{\mathcal{P}} &:= \tfrac{1}{2}(\langle\cdot,\cdot\rangle_{\mathcal{E}} + \langle\cdot,\cdot\rangle_{\mathcal{E}})\,, && (36)\\
\tau_{\mathcal{P}} &:= \tau_{\mathcal{E}} \otimes \tau_2\,, && (37)\\
\gamma_{\mathcal{P}} &:= \gamma_{\mathcal{E}} \otimes \mathbf{1}_2\,, && (38)\\
J_{\mathcal{P}} &:= J_{\mathcal{E}} \otimes \varepsilon_2\,, && (39)
\end{aligned}$$

The Pauli module:

$$\mathcal{V}_{\mathcal{P}} := \left\{ \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z} \end{pmatrix} \in \mathcal{P} \,|\, \mathfrak{z}^{\mathrm{cc}} = \mathfrak{z} \in \mathcal{E} \right\} \hookrightarrow \mathcal{P} \longrightarrow M\,, \tag{40}$$

whoses complexification $\mathcal{V}_{\mathcal{P}}^{\mathbb{C}}$ may be identified with the diagonal embedding

$$\mathcal{E} \hookrightarrow {}^2\mathcal{E}\,, \quad \mathfrak{z} \mapsto {}^2\mathfrak{z} \equiv \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z} \end{pmatrix}. \tag{41}$$

Proposition 2.1 *The most general real Dirac operator on* $\mathcal{P} = {}^2\mathcal{E}$ *reads:*

$$\not{D}_{\mathcal{P}} = \begin{pmatrix} \not{D}_{\mathcal{E}} & \phi_{\mathcal{E}} - \mathcal{F}_{\mathcal{E}} \\ \phi_{\mathcal{E}} + \mathcal{F}_{\mathcal{E}} & \not{D}_{\mathcal{E}}^{cc} \end{pmatrix} . \tag{42}$$

Here, respectively, $\not{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$ *is any Dirac operator on* $\mathcal{E} \longrightarrow M$ *and*

$$\begin{aligned} \phi_{\mathcal{E}}^{cc} &= +\phi_{\mathcal{E}} \, , \\ \mathcal{F}_{\mathcal{E}}^{cc} &= -\mathcal{F}_{\mathcal{E}} \end{aligned} \tag{43}$$

are general sections of $\mathrm{End}^+(\mathcal{E}) \longrightarrow M$*.*

Dirac operators of Pauli type: Let $\not{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$ be **real**.

$$\not{P}_{\mathrm{D}} := \begin{pmatrix} \not{D}_{\mathcal{E}} & -\mathcal{F}_{\mathcal{E}} \\ \mathcal{F}_{\mathcal{E}} & \not{D}_{\mathcal{E}} \end{pmatrix} \tag{44}$$

with $\mathcal{F}_{\mathcal{E}}$ being defined by the **(relative) curvature** of $\not{D}_{\mathcal{E}}$:

$$
\begin{aligned}
\mathcal{F}_{\mathcal{E}} := \not{\mathcal{F}}_{\mathrm{D}} &\equiv i\delta_\gamma(curv(\not{D}_{\mathcal{E}}) - \not{R}iem(g_{\mathrm{M}})) \\
&= i\not{F}_{\mathrm{D}} \,.
\end{aligned}
\tag{45}
$$

Note that $\not{F}_{\mathrm{D}} \in \mathfrak{S}ec(M, \mathrm{End}^+(\mathcal{E}))$ is even and real for real (or imaginary) Dirac operators $\not{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$. Whence, $\not{\mathcal{F}}_{\mathrm{D}}^{\mathrm{cc}} = -\not{\mathcal{F}}_{\mathrm{D}}$.

Definition 2.3 *Let $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}})$ be a real Clifford module bundle over (M, g_M) such that $\gamma_{\mathcal{E}}$ is real. Also, let $\mathcal{D}_{real}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E})$ be the (affine) set of real Dirac operators acting on $\mathfrak{S}ec(M, \mathcal{E})$.*

The **Pauli map** *is defined by*

$$
\begin{aligned}
\mathcal{P}_D : \mathcal{D}_{real}(\mathcal{E}) &\longrightarrow \mathcal{D}_{real}(\mathcal{P}) \\
\not{D}_{\mathcal{E}} &\mapsto \not{P}_D \,.
\end{aligned}
\tag{46}
$$

The two fermionic functionals:

$$\begin{aligned}
\mathcal{I}_{\mathrm{D,ferm}}: \mathfrak{S}ec(M,\mathcal{E}) \times \mathcal{D}(\mathcal{E}) &\longrightarrow \mathbb{C} \\
(\psi, D\!\!\!/_{\mathcal{E}}) &\mapsto \int_M \langle \psi, D\!\!\!/_{\mathcal{E}} \psi \rangle_{\mathcal{E}} \, dvol_{\mathrm{M}} \,, \qquad (47)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}'_{\mathrm{D,ferm}}: \mathfrak{S}ec(M,\mathcal{E}) \times \mathcal{D}(\mathcal{E}) &\longrightarrow \mathbb{C} \\
(\psi, D\!\!\!/_{\mathcal{E}}) &\mapsto \int_M \langle {}^2\psi, P\!\!\!/_{\mathrm{D}} {}^2\psi \rangle_{\mathcal{P}} \, dvol_{\mathrm{M}} \qquad (48)
\end{aligned}$$

contain the same information, actually:

However, a simple type Dirac operator

$$\begin{aligned}
D\!\!\!/_{\mathcal{E}} &= \partial\!\!\!/_{\mathrm{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathrm{D}} \\
&\neq D\!\!\!/^{\mathrm{cc}}_{\mathcal{E}} \qquad (49)
\end{aligned}$$

is not real, in general.

3 Dirac modules, the Pauli map and the STM action

Let $\mathcal{S} \longrightarrow M$ be a Dirac module and

$$\mathcal{E} := {}^2\mathcal{S} = \mathcal{S} \otimes \mathbb{C}^2 , \tag{50}$$

$$\langle \cdot, \cdot \rangle_{\mathcal{E}} := \tfrac{1}{2}(\langle \cdot, \cdot \rangle_{\mathcal{S}} + \langle \cdot, \cdot \rangle_{\mathcal{S}}) , \tag{51}$$

$$\tau_{\mathcal{E}} := \tau_{\mathcal{S}} \otimes \tau_2 , \tag{52}$$

$$\gamma_{\mathcal{E}} := \begin{pmatrix} \gamma_{\mathcal{S}} & 0 \\ 0 & \gamma_{\mathcal{S}}^{\mathrm{cc}} \end{pmatrix} , \tag{53}$$

$$J_{\mathcal{E}} := J_{\mathcal{S}} \otimes \varepsilon_2 . \tag{54}$$

Theorem 1 *The most general* **real Dirac operator of simple type**, *acting on* $\mathfrak{S}ec(M,\mathcal{E}={}^2\mathcal{S})$, *explicitly reads:*

$$\not{D}_{\mathcal{E}} \quad = \quad \not{\partial}_{\mathcal{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}}\,, \tag{55}$$

whereby $\not{\partial}_{\mathcal{A}} := \not{\partial}_{A} \oplus \not{\partial}_{A}^{cc}$ *is the real form of* $\not{\partial}_{A}$ *and*

$$\phi_{\mathcal{E}} := \begin{pmatrix} \chi_{\mathcal{S}} & \pm\phi_{\mathcal{S}}^{cc} \\ -\phi_{\mathcal{S}} & \mp\chi_{\mathcal{S}}^{cc} \end{pmatrix}, \tag{56}$$

depending on $\tau_{\mathcal{S}}^{cc} = \pm\tau_{\mathcal{S}}$, $\phi_{\mathcal{S}} \in \mathfrak{S}ec(M, \mathrm{End}^{+}_{\gamma}(\mathcal{S}))$ *reads:*

$$\phi_{\mathcal{S}} \equiv \begin{cases} \chi'_{\mathcal{S}} + \tau_{\mathcal{S}} \circ \delta_{\gamma}(\sigma_{\mathcal{S}})\,, & \textit{for} \quad \gamma_{\mathcal{S}}^{cc} = +\gamma_{\mathcal{S}}\,, \\ \underline{\tau_{\mathcal{S}} \circ \mu_{M}} + \delta_{\gamma}(\sigma_{\mathcal{S}})\,, & \textit{for} \quad \gamma_{\mathcal{S}}^{cc} = -\gamma_{\mathcal{S}}\,, \end{cases} \tag{57}$$

with μ_M, $\chi'_{\mathcal{S}} \in \Omega^0(M, \mathrm{End}^{+}_{\gamma}(\mathcal{S}))$, $\chi_{\mathcal{S}} \in \Omega^0(M, \mathrm{End}^{-}_{\gamma}(\mathcal{S}))$ *and* $\sigma_{\mathcal{S}} \in \Omega^1(M, \mathrm{End}^{-}_{\gamma}(\mathcal{S}))$.

Theorem 2 *Let $\mathcal{W} \longrightarrow M$ be a Majorana module, such that $\gamma_{\mathcal{W}}^{cc} = -\gamma_{\mathcal{W}}$. Also, let $\partial_{YMH} \in \mathcal{A}(\mathcal{W})$ be a Yang-Mills-Higgs connection and $\not{\partial}_{YMH} = \delta_\gamma \circ \partial_{YMH} = \not{\partial}_A + i\varphi_D$. Consider the real Dirac operator of simple type, called* **Dirac-Yukawa-Majorana operator***:*

$$\begin{aligned} \not{D}_{YM} &:= \begin{pmatrix} \not{\partial}_A + i\mu_D & i\mu_M \\ -i\mu_M & (\not{\partial}_A + i\mu_D)^{cc} \end{pmatrix} \\ &\equiv \not{\partial}_{\mathcal{A}} + i\mu_{YM} \in \mathcal{D}_{real}(\mathcal{E})\,. \end{aligned} \tag{58}$$

Here, respectively,

$$\not{\partial}_{\mathcal{A}} := \not{\partial}_A \oplus \not{\partial}_A^{cc} \in \mathcal{D}_{real}(\mathcal{E}) \tag{59}$$

is the real form of $\not{\partial}_A \in \mathcal{D}(\mathcal{S})$ and the **Majorana mass operator** *$\mu_M \in \Omega^0(M, \mathrm{End}_\gamma^+(\mathcal{S}))$ is real and $\mu_D \in \Omega^0(M, \mathrm{End}_\gamma^-(\mathcal{S}))$ is the* **Dirac**

mass operator *that is defined by the simple type Dirac operator*

$$\partial\!\!\!/_{A} + i\mu_{D} \equiv \begin{pmatrix} 0 & \partial\!\!\!/_{A} - i\varphi_{D} \\ \partial\!\!\!/_{A} + i\varphi_{D} & 0 \end{pmatrix} \in \mathcal{D}(\mathcal{S})\,. \tag{60}$$

The Euler-Lagrange equations of the **fermionic part** *of the Dirac action:*

$$\int_M \big(\langle^2\psi, \mathcal{P}_{D}(D\!\!\!\!/_{YM})^2\psi\rangle_{\mathcal{P}} + \mathrm{tr}_\gamma\big(curv(\mathcal{P}_{D}(D\!\!\!\!/_{YM})) - \varepsilon\, ev_g(\omega_{D}^2)\big)\big)\, dvol_{M} \tag{61}$$

read:

$$i\partial\!\!\!/_{A}\psi = \mu_{D}\psi + \mu_{M}\psi^{cc}\,, \tag{62}$$

$$(i\partial\!\!\!/_{A}\psi)^{cc} = \mu_{D}^{cc}\psi^{cc} + \mu_{M}\psi\,. \tag{63}$$

When restricted to $\tau_S\psi = \psi$, these equations become equivalent to:

$$
\begin{aligned}
i\partial\!\!\!/_{A}\chi &= \varphi_{D}\chi + m_{M}\chi^{cc} \qquad \Leftrightarrow \qquad \begin{cases} i\partial\!\!\!/\nu &= m_{D,\nu}\nu + m_{M,\nu}\nu^{cc} \\ i\partial\!\!\!/_{A} e &= \varphi_{D,e} e \end{cases} \qquad (64)\\
(i\partial\!\!\!/_{A}\chi)^{cc} &= \varphi_{D}^{cc}\chi^{cc} + m_{M}\chi\,, \qquad (65)
\end{aligned}
$$

whereby $\chi = (\nu, e) \in \mathfrak{S}ec(M, \mathcal{W} = \mathcal{W}_{\nu} \oplus \mathcal{W}_{e})$ and the splitting is defined by the **kernel of the real and constant Majorana mass operator** $m_{M} \in \mathfrak{S}ec(M, \mathrm{End}_{\gamma}^{+}(\mathcal{W}))$.

The **bosonic part** *of the Dirac action (61) reads:*

$$
\begin{aligned}
\int_M \big[&\mathrm{tr}_{\gamma}(curv(\mathcal{P}_{D}(\partial\!\!\!/_{\mathcal{A}}))) + a_1\,\mathrm{tr}_g(F_{\mathcal{A}}^2) + a_2\,\varepsilon\,\mathrm{tr}_g(\partial_{\mathcal{A}}\,\mu_{YM})^2 \\
&- a_2\,\mathrm{tr}_{\varepsilon}(\mu_{YM}^4) - a_4\,\mathrm{tr}_{\varepsilon}(\mu_{YM}^2)\big]\, dvol_{M} \qquad (66)
\end{aligned}
$$

with $a_1 = (n-3),\ a_2 = 2(n-2)(\frac{n-1}{n})^2,\ a_3 = 2\frac{(n-1)^3}{n^2},\ a_4 = 2\,.$

Furthermore,

$$\mathrm{tr}_g(\partial_{\mathcal{A}}\,\mu_{_{YM}})^2 \quad = \quad -4Re\,\mathrm{tr}_g(\partial_{_A}\varphi_{_{D,e}})^2\,, \tag{67}$$

$$a\,\mathrm{tr}_{\mathcal{E}}\mu^4_{_{YM}} + \mathrm{tr}_{\mathcal{E}}\mu^2_{_{YM}} \quad = \quad 4\mathrm{Re}\left(a\,\mathrm{tr}_{\mathcal{W}_e}\varphi^4_{_{D,e}} - \mathrm{tr}_{\mathcal{W}_e}\varphi^2_{_{D,e}} + \Lambda_{_{DM,\nu}}\right)\,, \tag{68}$$

whereby $a \equiv 2\,\frac{(n-1)^3}{n^2}$ *and*

$$\begin{aligned}\Lambda_{_{DM,\nu}} \quad \equiv \quad & a\,\mathrm{tr}_{\mathcal{W}_\nu}m^4_{_{D,\nu}} - \mathrm{tr}_{\mathcal{W}_\nu}m^2_{_{D,\nu}} + a\,\mathrm{tr}_{\mathcal{W}_\nu}m^4_{_{M,\nu}} - \mathrm{tr}_{\mathcal{W}_\nu}m^2_{_{M,\nu}} \\ & - 2a\,\mathrm{tr}_{\mathcal{W}_\nu}(m_{_{D,\nu}}\circ m_{_{M,\nu}})^2\end{aligned} \tag{69}$$

is the "true cosmological constant", which naturally occurs in the Einstein-Hilbert action when Majorana masses are taken into account within the geometrical frame of Dirac type gauge theories.

Conclusion:

"BE WISE, DO It TWICE" (at least)

Thank you!