

# Projectively equivalent pseudo-Riemannian metrics and integrable $so(p, q)$ -tops

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**Lorentzian Geometry**  
**Analysis and Geometry of**  
**pseudo-Riemannian Manifolds**

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- ▶ independent almost everywhere,
- ▶  $n = \frac{1}{2}(\dim M + \text{corank } A)$ .

Consider a finite-dimensional Lie algebra  $\mathfrak{g}$  and its dual space  $\mathfrak{g}^*$ .

## Definition

The Poisson-Lie bracket on  $\mathfrak{g}^*$  is defined by:

$$\{f, g\}(x) = \langle x, [df(x), dg(x)] \rangle.$$

Thus, each function  $H : \mathfrak{g}^* \rightarrow \mathbb{R}$  generates a Hamiltonian vector field on  $\mathfrak{g}^*$  which has a natural interpretation in terms of the coadjoint representation:

$$X_H(x) = \text{ad}_{dH(x)}^* x.$$

Complete Liouville integrability means that the corresponding Hamiltonian system ([Euler equation](#))

$$\dot{x} = X_H(x), \quad \dot{x}^i = c_{jk}^i x^j \frac{\partial H}{\partial x_k}$$

admits sufficiently many independent commuting first integrals  $f_1, \dots, f_n$ . The number  $n$  must be equal to  $\frac{1}{2}(\dim \mathfrak{g} + \text{ind } \mathfrak{g})$ , where  $\text{ind } \mathfrak{g}$  is a corank of the Poisson-Lie bracket at a generic point  $x \in \mathfrak{g}^*$ .

If  $\mathfrak{g}$  is semisimple then it admits an invariant form (Killing form) which allows us to identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$  and  $\text{ad}$  with  $\text{ad}^*$ . The Euler equation on  $\mathfrak{g}$  obtains the Lax form

$$\dot{x} = [dH(x), x].$$

**Important particular case:** quadratic Hamiltonians  $H(x) = \frac{1}{2}\langle R(x), x \rangle$  where  $R : \mathfrak{g} \rightarrow \mathfrak{g}$  is a symmetric operator. The Euler equation becomes

$$\dot{x} = [R(x), x]. \tag{1}$$

**Problem:** Describe/classify operators  $R : \mathfrak{g} \rightarrow \mathfrak{g}$  for which (1) is completely integrable.

Here  $\mathfrak{g} = so(n)$  is the Lie algebra of skew symmetric matrices.  
Assume that  $R : so(n) \rightarrow so(n)$  satisfies the following identity

$$[R(x), a] = [x, b], \quad x \in so(n), \quad (2)$$

for symmetric matrices  $a \neq 0$  and  $b$ . Then the following statement holds

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### Theorem (Manakov, Mischenko, Fomenko)

Let  $R : so(n) \rightarrow so(n)$  be symmetric and satisfy (2). Then

- ▶ the system (1) admits the following Lax representation with a parameter:

$$\frac{d}{dt}(x + \lambda a) = [R(x) + \lambda b, x + \lambda a];$$

- ▶ the functions  $\text{Tr}(x + \lambda a)^k$  are first integrals of (1) for any  $\lambda \in \mathbb{R}$  and, moreover, these integrals commute;
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- ▶ if  $a$  is regular, then (1) is completely integrable.

**Important remark:** From the algebraic point of view there is no difference between  $so(n)$  and  $so(p, q)$ .

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Instead of  $\bar{g}$ , it is convenient to introduce a linear operator ((1,1)-tensor):

$$L = \left( \frac{\det \bar{g}}{\det g} \right)^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

$L$  is (pseudo) self-adjoint w.r.t. both  $g$  and  $\bar{g}$ . Notice:  $\bar{g} = \frac{1}{\det L} g L^{-1}$ .

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## Theorem (classical result)

$g$  and  $\bar{g}$  are geodesically equivalent if and only if  $L$  satisfies the following equation:

$$\nabla_u L = \frac{1}{2} (u \otimes d \operatorname{tr} L + (u \otimes d \operatorname{tr} L)^*)$$

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for any vector field  $u$ . Or for those who likes "indices":

$$2L_{ij,k} = (\operatorname{tr} L)_{,i} g_{jk} + (\operatorname{tr} L)_{,j} g_{ik}.$$

## Compatibility conditions

For the equation  $\nabla_u L = F(u, L)$ , we compute:

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In our case:

$$R(u \wedge v)L - LR(u \wedge v) = (u \wedge v) \cdot M + ((u \wedge v) \cdot M)^*,$$

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Using  $g$ , we may think of  $u \wedge v$  as a skew-symmetric operator and of  $M$  as a symmetric operator. Then taking into account that

$$((u \wedge v) \cdot M)^* = M^* \cdot (u \wedge v)^* = -M \cdot (u \wedge v),$$

we have:

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## Theorem (Matveev, AB)

*If  $g$  admits a non-trivial geodesically equivalent partner  $\bar{g}$ , then the Riemann curvature tensor of  $g$  is a Manakov–Mischenko–Fomenko operator on  $so(g)$ .*

Thus, we have

$$[R(X), L] = [X, M].$$

where  $R$  is the curvature tensor,  $L$  is the operator which "connects"  $g$  and  $\bar{g}$ , and  $M$  is the Hessian of  $\text{Tr } L$ , and  $X$  is an arbitrary skew-symmetric operator.

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### Corollary

*If the curvature tensor of a given metric is not a MMF operator, then  $g$  admits no geodesically equivalent  $\bar{g}$ .*

## Theorem (Kiosak, Matveev, AB)

*Let  $\dim \geq 3$ . Assume that  $g$  admits two "independent" metrics  $g_1$  and  $g_2$  geodesically equivalent to it. Let  $g$  and  $g_1$  be strictly non-proportional. Then  $g$ ,  $g_1$  and  $g_2$  are all of constant curvature.*

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**Algebraic reformulation:** Let

$$[R(X), L_1] = [X, M_1] \quad \text{and} \quad [R(X), L_2] = [X, M_2],$$

where  $L_1$ ,  $L_2$  and  $Id$  are linearly independent. If  $L_1$  is regular, then  $R : so(n) \rightarrow so(n)$  is a scalar operator, i.e.,  $R(X) = k \cdot X$ .

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### Lemma

If  $[R(X), L_1] = [X, M_1]$  and  $[R(X), L_2] = [X, M_2]$ , then  $L_1$  is proportional either to  $M_1$ , or to  $L_2$ .

### Lemma

If  $L_1$  is regular and  $M_1 = k \cdot L_1$ , then  $R(X) = k \cdot X$ .

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$$\begin{aligned}\langle [[L_2, Y], M_1], Z \rangle &= \langle [R([L_2, Y]), L_1], Z \rangle = \langle R([L_2, Y]), [L_1, Z] \rangle = \langle [L_2, Y], R([L_1, Z]) \rangle \\ &= \langle Y, [R([L_1, Z]), L_2] \rangle = \langle Y, [[L_1, Z], M_2] \rangle = \langle [[M_2, Y], L_1], Z \rangle\end{aligned}$$

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The main identity (3) becomes:

$$YT + TY = M_2YL_1 + L_1YM_2 - M_1YL_2 - L_2YM_1,$$

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Taking "trace", we get:

$$(n+2)T + \text{Tr } T \cdot \text{Id} = M_2L_1 + L_1M_2 - L_2M_1 - M_1L_2 = 2T$$

Thus,  $T = 0$  and we come to a very simple identity:

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which holds for any symmetric matrix  $Y$ . We want to show that

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In some sense, this is a "matrix" analog of the following "vector" question:

Let

$$C = ml^T + lm^T$$

where  $m$  and  $l$  are vector-columns. Can we reconstruct  $l$  and  $m$  for a given  $C$ ?

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which holds for any symmetric matrix  $Y$ . We want to show that

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- ▶ or  $L_1$  is proportional to  $L_2$  and  $M_1$  is proportional to  $M_2$ .

In some sense, this is a "matrix" analog of the following "vector" question:

Let

$$C = ml^T + lm^T$$

where  $m$  and  $l$  are vector-columns. Can we reconstruct  $l$  and  $m$  for a given  $C$ ?

The answer is absolutely clear: yes, up to proportionality and permutation.

## Lemma

If  $L_1$  is regular and  $M_1 = k \cdot L_1$ , then  $R(X) = k \cdot X$ .

**Proof.**  $[R(X), L_1] = [X, M_1] \quad \Rightarrow \quad [R(X) - k \cdot X, L_1] = 0.$

If  $L_1$  is regular, then it is well known that its centralizer is generated by its powers  $(L_1)^k$ . In particular, the centralizer consists of symmetric operators only.

Since  $R(X) - k \cdot X$  is skew-symmetric, we obtain

$$R(X) = k \cdot X \quad \text{for any } X$$

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## THE CURVATURE IS CONSTANT