

Projectively equivalent pseudo-Riemannian metrics and integrable $so(p, q)$ -tops

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Lorentzian Geometry
Analysis and Geometry of
pseudo-Riemannian Manifolds

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- ▶ independent almost everywhere,
- ▶ $n = \frac{1}{2}(\dim M + \text{corank } A)$.

Consider a finite-dimensional Lie algebra \mathfrak{g} and its dual space \mathfrak{g}^* .

Definition

The Poisson-Lie bracket on \mathfrak{g}^* is defined by:

$$\{f, g\}(x) = \langle x, [df(x), dg(x)] \rangle.$$

Thus, each function $H : \mathfrak{g}^* \rightarrow \mathbb{R}$ generates a Hamiltonian vector field on \mathfrak{g}^* which has a natural interpretation in terms of the coadjoint representation:

$$X_H(x) = \text{ad}_{dH(x)}^* x.$$

Complete Liouville integrability means that the corresponding Hamiltonian system ([Euler equation](#))

$$\dot{x} = X_H(x), \quad \dot{x}^i = c_{jk}^i x^j \frac{\partial H}{\partial x_k}$$

admits sufficiently many independent commuting first integrals f_1, \dots, f_n . The number n must be equal to $\frac{1}{2}(\dim \mathfrak{g} + \text{ind } \mathfrak{g})$, where $\text{ind } \mathfrak{g}$ is a corank of the Poisson-Lie bracket at a generic point $x \in \mathfrak{g}^*$.

If \mathfrak{g} is semisimple then it admits an invariant form (Killing form) which allows us to identify \mathfrak{g} with \mathfrak{g}^* and ad with ad^* . The Euler equation on \mathfrak{g} obtains the Lax form

$$\dot{x} = [dH(x), x].$$

Important particular case: quadratic Hamiltonians $H(x) = \frac{1}{2}\langle R(x), x \rangle$ where $R : \mathfrak{g} \rightarrow \mathfrak{g}$ is a symmetric operator. The Euler equation becomes

$$\dot{x} = [R(x), x]. \tag{1}$$

Problem: Describe/classify operators $R : \mathfrak{g} \rightarrow \mathfrak{g}$ for which (1) is completely integrable.

Here $\mathfrak{g} = so(n)$ is the Lie algebra of skew symmetric matrices.
Assume that $R : so(n) \rightarrow so(n)$ satisfies the following identity

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Theorem (Manakov, Mischenko, Fomenko)

Let $R : so(n) \rightarrow so(n)$ be symmetric and satisfy (2). Then

- ▶ the system (1) admits the following Lax representation with a parameter:

$$\frac{d}{dt}(x + \lambda a) = [R(x) + \lambda b, x + \lambda a];$$

- ▶ the functions $\text{Tr}(x + \lambda a)^k$ are first integrals of (1) for any $\lambda \in \mathbb{R}$ and, moreover, these integrals commute;
- ▶ if a is regular, then (1) is completely integrable.

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Important remark: From the algebraic point of view there is no difference between $so(n)$ and $so(p, q)$.

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Instead of \bar{g} , it is convenient to introduce a linear operator ((1,1)-tensor):

$$L = \left(\frac{\det \bar{g}}{\det g} \right)^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

L is (pseudo) self-adjoint w.r.t. both g and \bar{g} . Notice: $\bar{g} = \frac{1}{\det L} g L^{-1}$.

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Theorem (classical result)

g and \bar{g} are geodesically equivalent if and only if L satisfies the following equation:

$$\nabla_u L = \frac{1}{2} (u \otimes d \operatorname{tr} L + (u \otimes d \operatorname{tr} L)^*)$$

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for any vector field u . Or for those who likes "indices":

$$2L_{ij,k} = (\operatorname{tr} L)_{,i} g_{jk} + (\operatorname{tr} L)_{,j} g_{ik}.$$

Compatibility conditions

For the equation $\nabla_u L = F(u, L)$, we compute:

$$\begin{aligned} \nabla_u \nabla_v L - \nabla_v \nabla_u L &= \\ \nabla_u F(v, L) - \nabla_v F(u, L) \end{aligned}$$

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In our case:

$$R(u \wedge v)L - LR(u \wedge v) = (u \wedge v) \cdot M + ((u \wedge v) \cdot M)^*,$$

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Using g , we may think of $u \wedge v$ as a skew-symmetric operator and of M as a symmetric operator. Then taking into account that

$$((u \wedge v) \cdot M)^* = M^* \cdot (u \wedge v)^* = -M \cdot (u \wedge v),$$

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Theorem (Matveev, AB)

If g admits a non-trivial geodesically equivalent partner \bar{g} , then the Riemann curvature tensor of g is a Manakov–Mischenko–Fomenko operator on $so(g)$.

Thus, we have

$$[R(X), L] = [X, M].$$

where R is the curvature tensor, L is the operator which "connects" g and \bar{g} , and M is the Hessian of $\text{Tr } L$, and X is an arbitrary skew-symmetric operator.

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Corollary

If the curvature tensor of a given metric is not a MMF operator, then g admits no geodesically equivalent \bar{g} .

Theorem (Kiosak, Matveev, AB)

Let $\dim \geq 3$. Assume that g admits two "independent" metrics g_1 and g_2 geodesically equivalent to it. Let g and g_1 be strictly non-proportional. Then g , g_1 and g_2 are all of constant curvature.

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Algebraic reformulation: Let

$$[R(X), L_1] = [X, M_1] \quad \text{and} \quad [R(X), L_2] = [X, M_2],$$

where L_1, L_2 and Id are linearly independent. If L_1 is regular, then $R : so(n) \rightarrow so(n)$ is a scalar operator, i.e., $R(X) = k \cdot X$.

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Lemma

If $[R(X), L_1] = [X, M_1]$ and $[R(X), L_2] = [X, M_2]$, then L_1 is proportional either to M_1 , or to L_2 .

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Let Y and Z be arbitrary symmetric matrices, We substitute $X = [L_2, Z]$ into $[R(X), L_1] = [X, M_1]$ and take "inner product" with Z :

$$\begin{aligned}\langle [[L_2, Y], M_1], Z \rangle &= \langle [R([L_2, Y]), L_1], Z \rangle = \langle R([L_2, Y]), [L_1, Z] \rangle = \langle [L_2, Y], R([L_1, Z]) \rangle \\ &= \langle Y, [R([L_1, Z]), L_2] \rangle = \langle Y, [[L_1, Z], M_2] \rangle = \langle [[M_2, Y], L_1], Z \rangle\end{aligned}$$

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The main identity (3) becomes:

$$YT + TY = M_2YL_1 + L_1YM_2 - M_1YL_2 - L_2YM_1,$$

where $T = M_2L_1 - L_2M_1 = L_1M_2 - M_1L_2$.

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Taking "trace", we get:

$$(n+2)T + \text{Tr } T \cdot \text{Id} = M_2L_1 + L_1M_2 - L_2M_1 - M_1L_2 = 2T$$

Thus, $T = 0$ and we come to a very simple identity:

$$M_2 Y L_1 + L_1 Y M_2 = M_1 Y L_2 + L_2 Y M_1$$

which holds for any symmetric matrix Y . We want to show that

- ▶ either L_1 is proportional to M_1 and M_2 is proportional to L_1 ,
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$$C = ml^T + lm^T$$

where m and l are vector-columns. Can we reconstruct l and m for a given C ?

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where m and l are vector-columns. Can we reconstruct l and m for a given C ?

The answer is absolutely clear: yes, up to proportionality and permutation.

Lemma

If L_1 is regular and $M_1 = k \cdot L_1$, then $R(X) = k \cdot X$.

Proof. $[R(X), L_1] = [X, M_1] \quad \Rightarrow \quad [R(X) - k \cdot X, L_1] = 0.$

If L_1 is regular, then it is well known that its centralizer is generated by its powers $(L_1)^k$. In particular, the centralizer consists of symmetric operators only.

Since $R(X) - k \cdot X$ is skew-symmetric, we obtain

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