Half-flat Structures and Special Holonomy

Vicente Cortés
University of Hamburg

Lorentzian Geometry. Analysis and Geometry of pseudo-Riemannian manifolds, Greifswald, 2009

This talk is based on joint work with Leistner, Schäfer and Schulte-Hengesbach: arXiv:0907.1222[math.DG]

Outline of the lecture:

- Half-flat structures
- Hitchin flow and special holonomy
- Examples

Compatible pairs of stable forms

- Let ω be a stable (i.e. non-degenerate) 2-form and ρ a stable 3-form on a six-manifold M
- $\Longrightarrow$ ρ induces an endomorphism field $J = J_\rho : TM \to TM$ such that $J^2 = \epsilon \mathbf{1}$, $\epsilon = \pm 1$.

Definition

The pair (ω, ρ) is called compatible if

$$\omega \wedge \rho = 0 \quad \text{and} \quad J^*\rho \wedge \rho = \frac{2}{3}\omega^3.$$

- Compatibility implies that J is skew-symmetric with respect to a pseudo-Riemannian metric g such that

$$\omega = g \circ J = g(\cdot, J\cdot).$$

Compatible pairs as H-structures

Proposition

A compatible pair (ω, ρ) defines an H-structure, where $H \subset GL(6, \mathbb{R})$ is any real form of $SL(3, \mathbb{C})$:

- $H = SU(p, q)$, if $\epsilon = -1$ and g has signature $(2p, 2q)$ $(p + q = 3)$,
- $H = SL(3, \mathbb{R})$, if $\epsilon = 1$, in which case g has signature $(3, 3)$.

Definition

An H-structure (ω, ρ) is called half-flat if

$$d\omega^2 = 0 \quad \text{and} \quad d\rho = 0.$$

- Half-flat $SU(3)$-structures have been introduced by Hitchin (Bilbao 2000) for the construction of metrics with holonomy G_2, the name is due to Chiossi and Salamon.

Special examples of half-flat $SU(3)$-structures

Strict nearly Kähler six-manifolds

- (M, g, J) strict NK $:\Longleftrightarrow g(JX, JY) = g(X, Y)$, $(\nabla_X J)Y = -(\nabla_Y J)X$ for all X, Y and $ker(X \mapsto \nabla_X J) = 0$.
- $\Longrightarrow$ half-flat with $\rho = \nabla\omega$ and

$$d\omega = 3\rho, \quad d(J^*\rho) = 2\omega^2.$$

- All known complete (strict) NK six-folds: $S^6,\ S^3 \times S^3,\ \mathbb{C}P^3,\ \mathbb{F}_{1,2}(\mathbb{C}^3)$.

Calabi-Yau 3-folds

- They satisfy $\nabla\omega = 0$ and

$$d\omega = 0, \quad d\rho = d(J^*\rho) = 0.$$

- Standard example: Quintic in $\mathbb{C}P^4$.

Proper examples of half-flat $SU(3)$-structures

- Main source of examples: Left-invariant structures on Lie groups.
- Taking the quotient of noncompact Lie groups by a lattice yields compact inhomogeneous examples,
- such as the Chiossi-Fino examples on certain nilmanifolds.
- There are also classification results, such as:

Theorem (Schulte-Hengesbach)

A direct product $G = G_1 \times G_2$ of three-dimensional Lie groups admits a left-invariant half-flat $SU(3)$-structure if and only if

(i) $\mathfrak{g} = \mathit{Lie}\, G$ *is unimodular or*

(ii) $\mathfrak{g}$ *is not solvable or*

(iii) $\mathfrak{g}$ *is isomorphic to* $\mathfrak{e}(2) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ *or to* $\mathfrak{e}(1,1) \oplus \mathfrak{r}_2 \oplus \mathbb{R}$.

- In particular, $S^3 \times S^3$ admits a left-invariant half-flat $SU(3)$-structure (other than the NK structure).

Evolution of a half-flat structure under the Hitchin flow

Theorem

- *Let* (ω_0, ρ_0) *be a half-flat H-structure on a six-manifold M, where H is any real form of* $SL(3, \mathbb{C})$.
- *Let* (ω, ρ) *be a pair of stable forms depending on a parameter* $t \in I$ *such that* $\omega(0) = \omega_0$, $\rho(0) = \rho_0$ *and*

$$\frac{\partial \rho}{\partial t} = d\omega, \quad \frac{\partial \hat{\omega}}{\partial t} = d\hat{\rho},$$

  where $\hat{\omega} = \frac{\omega^2}{2}$, $\hat{\rho} = J^* \rho$.
- *Then* $(\omega(t), \rho(t))$ *is a half-flat H-structure for all* $t \in I$ *and*
- $\varphi = \omega \wedge dt + \rho$ *defines a parallel G-structure on* $N = M \times I$, *where* $G = G_2$ *if* $H = SU(3)$ *and* $G = G_2^*$ *otherwise.*
- *The underlying Ricci-flat metric with holonomy in G is* $g_\varphi = g(t) - \epsilon dt^2$, *where* $g(t) = \epsilon \omega(t) \circ J_{\rho(t)}$.

  (For compact M and H the theorem is due to Hitchin.)

Existence

Corollary

- *Let M be a real analytic six-manifold with an analytic half-flat H-structure (ω_0, ρ_0).*
- *Then there exists a unique maximal solution (ω, ρ) of the Hitchin equations with initial data (ω_0, ρ_0), defined on an open neigborhood $\Omega \subset M \times \mathbb{R}$ of $M \times \{0\}$.*
- *In particular, there is a parallel $G_2^{(*)}$-structure on Ω.*
- *Any automorphism f of (ω_0, ρ_0) defines an automorphism of $(\omega(t), \rho(t))$ on $U_t = \{p \in M | (p, t) \in \Omega \text{ and } (f(p), t) \in \Omega\}$.*
- *If, in addition, M is compact or homogeneous with invariant H-structure, then $\Omega = M \times I$ for some interval I.*

Completeness

- The $G_2^{(*)}$-metric on $\Omega \subset M \times \mathbb{R}$ can only be geodesically complete if $\Omega = M \times \mathbb{R}$.
- If the metric is positive definite, that can only happen if it splits as a product (Cheeger-Gromoll).
- It is difficult to construct examples such that Ω can be isometrically embedded into a complete manifold with holonomy in $G_2^{(*)}$.
- The Bryant-Salamon G_2-metric on the total space of the spinor bundle of S^3 is such an example.
- Allowing *conformal* diffeomorphisms we have:

Corollary

- *Let M be a compact real analytic six-manifold with an analytic half-flat $SU(3)$-structure.*
- *Then there is complete Riemannian metric on $N = M \times \mathbb{R}$ which is globally conformal to the G_2-metric obtained from the Hitchin flow.*

Nearly half-flat structures and nearly parallel $G_2^{(*)}$-structures

Definition

- An H-structure (ω, ρ) is called **nearly half-flat** if

$$d\rho = \lambda\hat{\omega}$$

  for some constant $\lambda \neq 0$.

- A $G_2^{(*)}$-structure φ is called **nearly parallel** if

$$d\varphi = \mu *_{\varphi} \varphi$$

  for some constant $\mu \neq 0$.

Nearly half-flat structures and nearly parallel $G_2^{(*)}$-structures

Theorem

- *Let* (ω_0, ρ_0) *be a nearly half-flat H-structure.*
- *Let* (ω, ρ) *be a pair of stable forms depending on a parameter* $t \in I$ *with* $\omega(0) = \omega_0$, $\rho(0) = \rho_0$ *and*

$$\frac{\partial \rho}{\partial t} = d\omega + \lambda \hat{\rho}, \quad \frac{\partial \hat{\omega}}{\partial t} = d\hat{\rho}.$$

- *Then* $(\omega(t), \rho(t))$ *is a nearly half-flat H-structure for all* $t \in I$ *and*
- $\varphi = \omega \wedge dt + \rho$ *defines a nearly parallel G-structure on* $N = M \times I$, *where* $\mu = -\epsilon\lambda$, $G = G_2$ *if* $H = SU(3)$ *and* $G = G_2^*$ *otherwise.*

  (For compact M and H the theorem is due to Stock.)

Cocalibrated $G_2^{(*)}$-structures and parallel *Spin*(7)- and $Spin_0(3,4)$-structures

Definition

- A $G_2^{(*)}$-structure φ is called **cocalibrated** if

$$d *_\varphi \varphi = 0.$$

Cocalibrated $G_2^{(*)}$-structures and parallel $Spin(7)$- and $Spin_0(3,4)$-structures

Theorem

- *Let $I \ni t \mapsto \varphi(t)$ be a one-parameter family of stable 3-forms on a 7-manifold M*
- *satisfying the evolution equation*

$$\frac{\partial}{\partial t} *_\varphi \varphi = d\varphi.$$

- *If φ is a cocalibrated $G_2^{(*)}$-structure at $t = t_0 \in I$, then $\varphi(t)$ is a cocalibrated $G_2^{(*)}$-structure for all $t \in I$ and*
- *the 4-form*

$$\Phi = dt \wedge \varphi + *_\varphi \varphi$$

defines a parallel $Spin(7)$- or $Spin_0(3,4)$-structure on $N = M \times I$ with the metric $g_\Phi = g_\varphi + dt^2$.

(Due to Hitchin for compact M and φ of type G_2.)

Examples: Evolution of nearly ϵ-Kähler manifolds as half-flat manifolds

- Let (M, J_0, g_0) be a nearly ϵ-Kähler manifold with $|\nabla J_0|^2 = 4$.
- Then the H-structure $(\omega_0 = g_0 \circ J_0, \rho_0 = \frac{1}{3} d\omega_0)$ is half-flat.
- Its evolution is

$$\omega = t^2 \omega_0, \quad \rho = t^3 \rho_0 \quad (t_0 = 1)$$

- and the corresponding metric with holonomy in $G_2^{(*)}$ on $N = M \times (0, \infty)$ is the cone metric

$$\hat{g}_0 = t^2 g_0 - \epsilon dt^2.$$

- Similarly, the evolution of a nearly parallel $G_2^{(*)}$-structure (with $\mu = 4$) yields the cone metric in 8 dimensions.

Examples: Evolution of nearly ϵ-Kähler manifolds as nearly half-flat manifolds

- Let (M, J_0, g_0) be a nearly ϵ-Kähler manifold with $|\nabla J_0|^2 = 4$.
- Then the H-structure $(\omega_0 = g_0 \circ J_0, -\hat{\rho}_0 = -\frac{1}{3}\widehat{d\omega_0})$ is nearly half-flat.
- Its evolution ($t_0 = 0$) is

$$\omega = c(t)^2 \omega_0, \quad \rho = -c(t)^3 (s(t)\rho_0 + c(t)\hat{\rho}_0),$$

  where $c(t) = \cos(t)$ for $\epsilon = -1$ and $c(t) = \cosh(t)$ for $\epsilon = 1$.
- The metric associated to the corresponding nearly parallel $G_2^{(*)}$-structure on $N = M \times I$ is

$$g_N = -\epsilon(c(t)^2 g_0 + dt^2),$$

  where $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ or $I = \mathbb{R}$, respectively.

Examples: Half-flat structures on nilmanifolds $\Gamma \setminus H_3 \times H_3$

Theorem (Classification of left-invariant 1/2-flat structures on $H_3 \times H_3$)

(i) *Any stable 2-form ω on $\mathfrak{l} := \mathfrak{h}_3 \oplus \mathfrak{h}_3$ such that $d\omega^2 = 0$ is equivalent, up to automorphisms of $\mathfrak{l}$ and rescalings, to one of 5 normal forms:*

$$\omega_1 = e^1 f^1 + e^2 f^2 + e^3 f^3, \ldots, \omega_5,$$

where ω_4 depends (affinely) on a real parameter.

(ii) *The closed stable 3-forms compatible with a multiple of ω_i are given by a (linear) 9-parameter family*

$$\rho = \rho_i(a_1, \ldots, a_9),$$

subject to a quartic nondegeneracy condition $\lambda(\rho) \neq 0$.

Half-flat $SU(3)$-structures on $H_3 \times H_3$

Theorem

Any left-inv. half-flat $SU(3)$-structure (ω, ρ) on $H_3 \times H_3$ is equivalent to

$\omega = \omega_1 = e^1 f^1 + e^2 f^2 + e^3 f^3$,

$\rho = \rho_1(a_1, \ldots, a_9) =$
$a_1\, e^{123} + a_2\, f^{123} + a_3\, e^1 f^{23} + a_4\, e^2 f^{13} + a_5\, e^{23} f^1 + a_6\, e^{13} f^2 +$
$a_7\, (e^2 f^{23} - e^1 f^{13}) + a_8\, (e^{12} f^3 - e^3 f^{12}) + a_9\, (e^{23} f^2 - e^{13} f^1)$,

subject to the inequality $g_{\omega,\rho} > 0$ ($\Longrightarrow \lambda(\rho) < 0$) and normalisation of ρ.

Examples of half-flat structures on $H_3 \times H_3$ with $\omega = \omega_1$

- $\rho = \frac{1}{\sqrt{2}}(e^{123} - f^{123} - e^1 f^{23} + e^2 f^{13} + e^{23} f^1 - e^{13} f^2 + e^{12} f^3 - e^3 f^{12})$ defines a half-flat $SU(3)$-structure for which the standard basis of $\mathfrak{l}$ is orthonormal.
- $\rho = \frac{1}{\sqrt{2}}(e^{123} - f^{123} - e^1 f^{23} - e^2 f^{13} + e^{23} f^1 + e^{13} f^2 - e^{12} f^3 + e^3 f^{12})$ defines a half-flat $SU(1,2)$-structure for which the standard basis of $\mathfrak{l}$ is orthonormal with spacelike e_1 and e_4.
- $\rho = \sqrt{2}(e^{123} + f^{123})$ defines a half-flat $SL(3,\mathbb{R})$-structure for which the factors of $H_3 \times H_3$ are isotropic.

Theorem

- *The above examples yield via the Hitchin flow explicit metrics of full holonomy G_2 (1st case) or G_2^*.*
- *Restricting $\rho(a_1, \ldots, a_9)$ to some neighborhood of the above examples yields a family of metrics of full holonomy $G_2^{(*)}$ depending on 8 (normalised) parameters.*

Half-flat structures on $H_3 \times H_3$ with $\omega \not\sim \omega_1$

Theorem

- *Let (ω, ρ) be a left-inv. half-flat structure on $M = H_3 \times H_3$ such that $\omega \not\sim \omega_1$.*
- *Then $(M, g = g_{\omega,\rho})$ is a product of the unique 4-dimensional s.c. non-flat para-hyper-Kähler symmetric space (S, g_S) and a 2-dimensional flat factor.*
- *The Hitchin flow is defined for all times and provides a metric with holonomy group in G_2^* on $N = M \times \mathbb{R}$.*
- *The metric is either flat or isometric to the product of g_S and a 3-dimensional flat metric.*

Special geometry of spaces of 3-forms in six dimensions

- Projective special Kähler manifolds are the target manifolds of $N = 2$ Einstein-Maxwell supergravity.
- $\exists$ Classification of homogeneous projective special pseudo-Kähler manifolds of semisimple groups with compact stabiliser: Alekseevsky and Cortés, Proc. London Math. Soc. 2000.
- $\implies \exists^1$ homogeneous projective special pseudo-Kähler manifold of a real form of $SL(6, \mathbb{C})$ with compact stabiliser:

$$\frac{SU(3,3)}{S(U(3) \times U(3))}.$$

- It arises as open orbit of $SU(3,3)$ on the highest weight orbit of $SL(6, \mathbb{C})$ in $P(\Lambda^3(\mathbb{C}^6)^*)$.
- In [CLSS] we extend this classification for groups of type A_5 by allowing noncompact stabilisers.

Theorem

The homogeneous projective special pseudo-Kähler manifolds $\bar{M} = G/H$ of real simple groups G of type A_5 are listed in the following table ($\dim_{\mathbb{C}} \bar{M} = 9$):

G/H	*Hermitian signature*
$SU(3,3)/S(U(3) \times U(3))$	*(0,9)*
$SU(3,3)/S(U(2,1) \times U(1,2))$	*(4,5)*
$SU(5,1)/S(U(3) \times U(2,1))$	*(6,3)*
$SL(6,\mathbb{R})/(U(1) \cdot SL(3,\mathbb{C}))$	*(3,6)*

In addition there is the following homogeneous projective special para-Kähler manifold

$$\frac{SL(6,\mathbb{R})}{S(GL(3,\mathbb{R}) \times GL(3,\mathbb{R}))}.$$

Each of these spaces is associated with an open orbit of $\mathbb{R}^* \cdot G$ on a real form of $\Lambda^3(\mathbb{C}^6)^*$.

Example (Hitchin)

$GL^+(6,\mathbb{R})$ has precisely two open orbits on $\Lambda^3(\mathbb{R}^6)^*$:

$$\begin{aligned}\{\lambda < 0\} &\cong \frac{GL^+(6,\mathbb{R})}{SL(3,\mathbb{C})},\\ \{\lambda > 0\} &\cong \frac{GL^+(6,\mathbb{R})}{SL(3,\mathbb{R})\times SL(3,\mathbb{R})}.\end{aligned}$$

- Notice that

$$J_\rho^2 = \begin{cases} -\mathbf{1} & \text{if} \quad \lambda(\rho) < 0 \\ +\mathbf{1} & \text{if} \quad \lambda(\rho) > 0. \end{cases}$$