International Summer School Greifswald, 2009 July 22

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Marc Nardmann

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"Relative h-principles" for negative scalar curvature. Are Lorentzian metrics of negative scalar curvature ubiquitous?

... a priori only loosely related, but there is a common underlying theme:

nonintegrable distributions. (*distribution on* M = sub vector bundle of TM)

§1. The dominant energy condition

… is an inequality for the Ricci curvature of a Lorentzian n-manifold.

Sign convention: *Lorentzian* means $-{+}{+}\cdots{+}$, i.e. index 1, signature $(n-1,1)$.

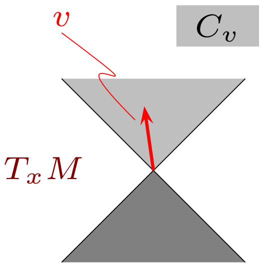
$v \in TM$ is *spacelike* $\Leftrightarrow g(v,v) > 0$

$v \in TM$ is *timelike* $\Leftrightarrow g(v,v) < 0$

$v \in TM$ is *lightlike* $\Leftrightarrow g(v,v) = 0$ and $v \neq 0$.

Definition. Let (M,g) be a Lorentzian manifold, let $\Lambda \in \mathbb{R}$.
Consider the *energy-momentum tensor* $T := \mathrm{Ric}_g - \frac{1}{2}\mathrm{scal}_g g + \Lambda g$ with respect to the "cosmological constant" Λ.

For a timelike or lightlike $v \in T_xM$, let C_v denote the closure of the connected component of $\{w \in T_xM \mid w \text{ is timelike or lightlike}\}$ which contains v.



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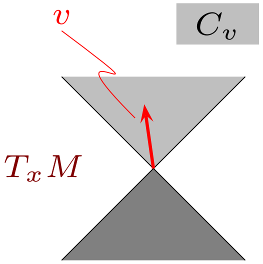

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(M,g) satisfies ***the dominant energy condition w.r.t.*** Λ iff $-T_a{}^b v^a \in C_v$ for every timelike (or lightlike) $v \in TM$.

(M,g) satisfies ***the strict dominant energy condition w.r.t.*** Λ iff $-T_a{}^b v^a \in \mathrm{interior}(C_v)$ for every timelike or lightlike $v \in TM$.

Does every manifold which admits a Lorentzian metric at all admit one which satisfies the dominant energy condition?

The answer is *yes* in most cases:

Theorem 1. Let (M, g) be a connected Lorentzian manifold of dimension $n \geq 4$, let $\Lambda \in \mathbb{R}$, let K be a compact subset of M.

If $n = 4$, assume that (M, g) is time- and space-orientable, and that either $\partial M \neq \varnothing$, or M is closed with intersection form signature divisible by 4.

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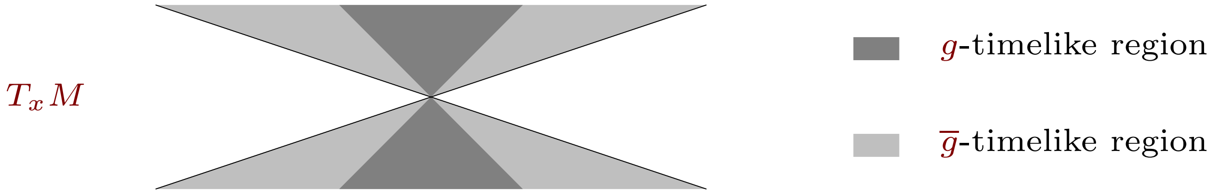
Then there exists a Lorentzian metric $\overline{g}$ *on* M *such that*

- $\overline{g}$ *satisfies the strict dominant energy condition with respect to* Λ *on* K;
- *every* g*-timelike or* g*-lightlike vector in* TM *is* $\overline{g}$*-timelike.*

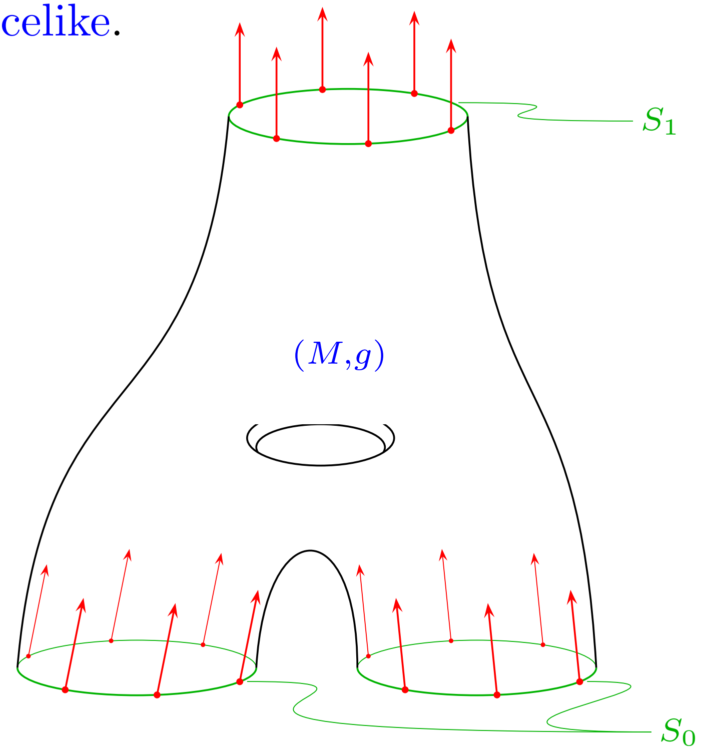


A classical topic in General Relativity is the problem of ***topology change***:

Definition. Let S_0, S_1 be $(n-1)$-dimensional closed manifolds.

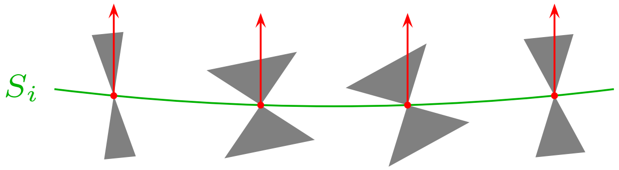
A ***weak Lorentz cobordism*** between S_0 and S_1 is a compact Lorentzian n-manifold (M, g) whose boundary is the disjoint union $S_0 \sqcup S_1$, such that M admits a timelike vector field which is inward-directed on S_0 and outward-directed on S_1.

A ***Lorentz cobordism*** is a weak Lorentz cobordism (M, g) such that ∂M is g-spacelike.

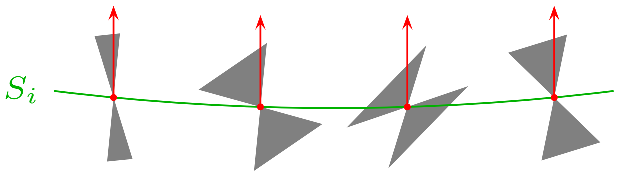


↑↑↑ timelike vector field on (M, g)

lightcones at the boundary:



Lorentz cobordism



weak Lorentz cobordism

Fact. S_0, S_1 are weakly Lorentz cobordant **iff** they are Lorentz cobordant.

Theorem [Tipler 1977]. *If there exists a Lorentz cobordism* (M, g) *between closed* 3*-manifolds* S_0 *and* S_1 *such that* $\mathrm{Ric}_g(v, v) > 0$ *holds for all lightlike* $v \in TM$, *then* S_0 *and* S_1 *are diffeomorphic.*

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Theorem 2. Let S_0, S_1 be arbitrary closed orientable 3-manifolds, let $\Lambda \in \mathbb{R}$. *Then there exists a* ***weak*** *Lorentz cobordism* (M, g) *between* S_0 *and* S_1 *which satisfies the strict dominant energy condition with respect to* Λ*; in particular,* $\mathrm{Ric}_g(v, v) > 0$ *holds for all lightlike* $v \in TM$.

§2. Prescribed scalar curvature

Well-known:
- Every manifold of dim. ≥ 3 admits a *Riemannian* g with $\mathrm{scal}_g < 0$.
- Some closed manifolds, e.g. the n-dimensional torus, do **not** admit a *Riemannian* metric g with $\mathrm{scal}_g > 0$.

Which smooth functions are scalar curvatures of *Lorentzian* metrics?
And what about pseudo-Riemannian metrics of index ≥ 2?

Convention. In the rest of this §2, we assume that
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all manifolds are connected and ***compact****, possibly with boundary.*

Theorem 3. Let (M, g) be a semi-Riemannian manifold of signature (p, q), where $p \geq 3$ and $q \geq 1$. Let $s \in C^\infty(M, \mathbb{R})$.
Then M admits a metric $\overline{g}$ of index q and scalar curvature s such that every g-timelike or g-lightlike vector in TM is $\overline{g}$-timelike.

(No restriction on s! Take e.g. (M, g) to be a flat Lorentzian n-torus and $s \equiv 1$.)

Let $\mathscr{M}_q(M)$ denote the set of semi-Riemannian metrics of index q on a manifold M. We equip $\mathscr{M}_q(M)$ (viewed as a set of sections in the bundle $\mathrm{Sym}^2 T^*M \to M$) with the compact-open topology.

Theorem 4. Let M be an n-manifold with $n \geq 3$, or $n = 2$ and $\partial M \neq \varnothing$. Let $1 \leq q \leq n-1$. *Then for* ***every*** $s \in C^\infty(M, \mathbb{R})$, ***every*** *connected component of* $\mathscr{M}_q(M)$ *contains a metric of scalar curvature* s.

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When $n = 2$ and $\partial M = \varnothing$, an obstruction occurs:

The **Gauß/Bonnet formula for closed Lorentzian surfaces [Avez 1962]** says that

$$\int_M \mathrm{scal}_g \, \mu_g = 0 \ .$$

So if $s = \mathrm{scal}_g$ for some metric g, then s is identically 0 or changes its sign.

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Theorem 5. Let M be the 2-dimensional torus or the Klein bottle (these are the only closed nonempty 2-manifolds which admit a Lorentzian metric). *A function* $s \in C^\infty(M, \mathbb{R})$ *is the scalar curvature of a Lorentzian metric on* M ***if*** *and only if* s *is either identically* 0 *or changes its sign.*

§3. Relative h-principles for scal < 0

(The convention from §2 is no longer valid: manifolds are not necessarily compact.)

Definition of the fine $C^{k,\alpha}$-topology.

Let $k \in \mathbb{N}$ and $\alpha \in [0,1]$.
We define the *fine $C^{k,\alpha}$-topology* on the set of C^∞ sections in $\mathrm{Sym}^2 T^*M \to M$:

Let h be a Riem. metric on M, let $K \subseteq (M,h)$ be compact & geodesically convex.

For $x, y \in K$, let $\tau_{x,y} :=$ parallel transport along the unique geodesic $x \to y$ in K.

We have the Hölder $C^{k,\alpha}$-norm $\|.\|_{C^{k,\alpha}(K;h)}$ for C^∞ tensor fields η, σ on K:

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$$D^j \eta \;:=\; \underbrace{\nabla^h \cdots \nabla^h}_{j \text{ derivatives}} \eta \qquad\qquad \|\sigma\|_{C^0(K;h)} \;:=\; \max\{|\sigma(x)|_h \mid x \in K\}$$

$$[\sigma]_{C^{0,\alpha}(K;h)} \;:=\; \sup\left\{ \frac{|\tau_{x,y}(\sigma(x)) - \sigma(y)|_h}{\mathrm{dist}_h(x,y)^\alpha} \;\middle|\; x, y \in K,\ x \neq y \right\}$$

$$\|\eta\|_{C^{k,\alpha}(K;h)} \;:=\; \textstyle\sum_{j=1}^k \|D^j\eta\|_{C^0(K;h)} + [D^k\eta]_{C^{0,\alpha}(K;h)}$$

The topology induced by $\|.\|_{C^{k,\alpha}(K;h)}$ does not depend on h.

The *fine $C^{k,\alpha}$-topology* on $\mathscr{M}_q(M)$ is defined via neighbourhood bases $\mathfrak{B}_g$:

We choose
- a Riemannian metric h on M;
- a locally finite family $(K_i)_{i\in\mathbb{N}}$ of compact & geodesically convex sets $\subseteq M$ whose interiors cover M.

For every sequence $\varepsilon = (\varepsilon_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ and every $g \in \mathscr{M}_q(M)$, we define

$$\begin{aligned} \mathscr{N}_{g,\varepsilon} &:= \left\{\overline{g} \in \mathscr{M}_q(M) \;\middle|\; \forall i \in \mathbb{N}\colon \|\overline{g} - g\|_{C^{k,\alpha}(K_i;h)} < \varepsilon_i\right\} \\ \mathfrak{B}_g &:= \left\{\mathscr{N}_{g,\varepsilon} \;\middle|\; \varepsilon = (\varepsilon_i)_{i\in\mathbb{N}},\ \varepsilon_i \in \mathbb{R}_{>0}\right\} \quad . \end{aligned}$$

The topology defined by the nbhd. bases $\mathfrak{B}_g$ does not depend on h or $(K_i)_{i\in\mathbb{N}}$.

In the Riemannian case $q = 0$, J. Lohkamp proved the following:

Theorem [Lohkamp 1995: the "relative C^0-dense h-principle for $\mathrm{scal} < c$"].
Let (M, g) be a Riemannian manifold of dimension $p \geq 3$.
Let A be a closed subset of M, let $c \in \mathbb{R}$ be such that $\mathrm{scal}_g < c$ holds on A.
Let $\mathscr{U}$ be a neighbourhood of g in $\mathscr{M}_0(M)$ with respect to the fine C^0-topology.

Then there exists a metric $\overline{g} \in \mathscr{U}$ with $\mathrm{scal}_{\overline{g}} < c$ such that $\overline{g} = g$ holds on A.

In particular (take $A = \varnothing$, $c = 0$), *the set of negative scalar curvature metrics is dense in the space of Riemannian metrics with respect to the fine C^0-topology.*

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(Lohkamp also proved an analogous theorem with Ric instead of scal.)
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Theorem 6 [the semi-Riem. relative $C^{0,\alpha}$-dense h-principle for $\mathrm{scal} < c$].
Let (M, g) be a semi-Riemannian manifold of signature (p, q), where $p \geq 3$.
Let $0 \leq \alpha < 1$. Let A be a closed subset of M, let $c \in \mathbb{R}$ with $\mathrm{scal}_g < c$ on A.
Let $\mathscr{U}$ be a neighbourhood of g in $\mathscr{M}_0(M)$ with respect to the fine $C^{0,\alpha}$-topology.

Then there exists a metric $\overline{g} \in \mathscr{U}$ with $\mathrm{scal}_{\overline{g}} < c$ such that $\overline{g} = g$ holds on A.

§4. The common theme of the proofs: “stretching” metrics along nonintegrable distributions

Techniques in the proofs that we will *not* discuss here:

- prescribed scalar curvature $\rightsquigarrow$ differential *equation* (underdetermined);
  trick: restrict underdetermined PDE to *elliptic* PDE
- Gromov’s h-principle for ample partial differential relations
  + elementary algebraic topology $\rightsquigarrow$ existence of nonintegrable distributions

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  + elementary algebraic topology ⇝ existence of nonintegrable distributions

Focus on *inequalities*:

- How to prove existence of positive scalar curvature metrics?
  Why does the proof fail in the Riemannian case?
- How to prove ubiquity of negative scalar curv. metrics (relative h-principle)?
- How to construct dominant energy metrics?

The basic construction is the same in each case:

stretching along a nonintegrable distribution.

Recall that a k-plane distribution H on a manifold M (i.e. a sub vector bundle of rank k of TM) is *integrable* iff one of the following equivalent statements holds:

1. There exists a k-dimensional foliation F of M such that $\forall x \in M\colon H_x = T_x F$.
2. $\forall v, w \in \Gamma(H) : \; [v, w] \in \Gamma(H)$.

We're interested in distributions H which are *nonintegrable* in the following sense:

$$\forall x \in M : \exists v, w \in \Gamma(H) : [v, w](x) \notin H_x \quad .$$

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let $TM = V \oplus H$ be a g-orthogonal splitting. We *"stretch"* $g = g_V \oplus g_H$ *along* V:

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The scalar curvature of $\overline{g}$ looks like this (with $q = \mathrm{index}(g)$):

$$\begin{aligned}\mathrm{scal}_{\overline{g}} = \mathrm{scal}_g &+ \tfrac{2q}{f}\Delta_g^H(f) + 2(q-1) f \Delta_g^V(f) - \tfrac{q(q+3)}{f^2}|df|^2_{g,H} - q(q-1)|df|^2_{g,V} \\ &+ 2(q-2) f \big\langle \mathrm{div}_g^H, df \big\rangle_{g,V} + \tfrac{2(q+1)}{f}\big\langle \mathrm{div}_g^V, df \big\rangle_{g,H} \\ &+ (1-f^2)\xi_g^V \ - \frac{f^4 - f^2}{4}\big|\mathcal{T}w_g^V\big|_g^2 \ - \frac{1-f^2}{4f^2}\big|\mathcal{T}w_g^H\big|_g^2 \quad .\end{aligned}$$

Dominant energy: Idea of proof.

1. Choose a g-orthogonal splitting $TM = V \oplus H$ with V timelike, H spacelike. Show that this can be done with **nonintegrable** H.
2. Using a very **small constant** $f > 0$, stretch g along V $\rightsquigarrow$ d.e.c. holds. If $f < 1$, then the lightcones of the new metric are wider than those of g.

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Relative $C^{0,\alpha}$-dense h-principle: Idea of proof.

1. Choose a g-orthogonal splitting $TM = V \oplus H$ with V timelike, H spacelike. Keep the metric fixed on V, modify it on H as follows.
2. Cover M by compact balls. On each ball B_i, choose a 1-parameter family of g-orth. splittings $H = V_c' \oplus H_c'$ s.t. H_c' becomes "more nonintegrable" as $c \to \infty$.
3. Choose cutoff $\beta \in C^\infty(M, [0,1])$ which is 0 outside B_i and is 1 on a slightly smaller ball B_i'. Using $f = 1 - a\beta \in C^\infty(M, \mathbb{R}_{>0})$ with a small constant $a > 0$, stretch g along V_c'.

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4. Do this with $c = c(a)$, s.t. $\lim_{a\to 0} c(a) = \infty$, but $c(.)$ not faster increasing than necessary. For $a \to 0$, the new metric is $C^{0,\alpha}$-close to g, but has scal $\ll 0$ on B'_i.
5. Do this for every B_i.

Prescribed scalar curvature: Idea of proof (very roughly).

1. Solve the elliptic equation (mentioned above, but not explained) via the method of sub- and supersolutions. The equation depends on data g and H. Construct g and H such that sub- and supersolutions exist, as follows.
2. Choose a splitting $TM = V \oplus H$ with V timelike, H spacelike.
   Show that this can be done with H being nonintegrable (almost everywhere).
3. Change the metric g on H as in the h-principle proof, creating so much negative curvature that (say) the constant 0.1 becomes a supersolution.
4. Because H is nonintegrable, sufficiently small constants $f < 1$ are subsolutions.
   (If H is nonintegrable only *almost* everywhere, deal with the problem set by making f slightly nonconstant near it.)

The elliptic equation

$$
\begin{aligned}
0 = &+2\Delta_g(f) + a_{n,q}(f)|df|_g^2 + b_{n,q}(f)|df|_{g,V}^2 \\
&+\frac{2(1+f^2)}{f^2}\langle \mathrm{div}_g^V, df\rangle_{g,H} + 2(1+f^2)\langle \mathrm{div}_g^H, df\rangle_{g,V} \\
&+\frac{(1+f^2)^2}{2f^3}|\mathrm{Twist}_H|_g^2 - \frac{f(1+f^2)^2}{2}|\mathrm{Twist}_V|_g^2 + \frac{(1+f^2)^2}{f}\xi_{g,V} \\
&+\frac{1+f^2}{f}\mathrm{scal}_g - f^{\frac{2q}{n-1}-1}(1+f^2)^{1-\frac{1}{n-1}} s \quad .
\end{aligned}
$$

The functions $a_{n,q}, b_{n,q} \in C^\infty(\mathbb{R}_{>0}, \mathbb{R})$ are defined by

$$
\begin{aligned}
a_{n,q}(x) &:= \frac{\big((q-1)^2-(n-1)(q+3)\big)x^4 - 2(q-1)(n-1-q)x^2 - q(n-1-q)}{(n-1)x^3(1+x^2)} \quad , \\
b_{n,q}(x) &:= \frac{(q-1)(n-q)x^4 + 2(q-1)(n-1-q)x^2 + q(n-1-q)}{(n-1)x^3} \quad .
\end{aligned}
$$

The first-order terms are essentially irrelevant.