Geometrical Aspects of Time Machines

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Outline

Preliminaries

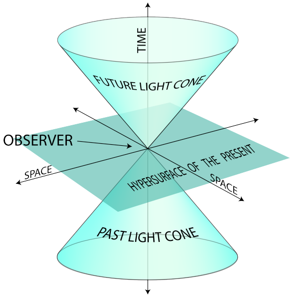
Causality of Conformally Stationary Spacetimes

Causality of Gödel-type Spacetimes

Causality of Other Generalizations of Gödel spacetime

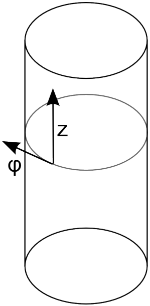
Notation

Spacetime: connected, time-oriented Lorentzian manifold (M, g) of signature $(-, +, \ldots, +)$, $\dim M = n + 1$



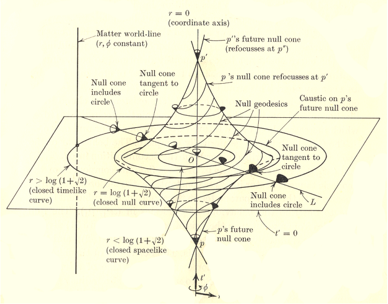
Non-causal/-chronological Spacetimes

"Trivial" CTCs (in the sense of Carter)



$M = S^1 \times \mathbb{R},\ g = -d\varphi^2 + dz^2$

"Non-trivial" CTCs



$M = \mathbb{R}^3,\ g = -dt^2 + dx^2 - \frac{1}{2} e^{2x} dy^2 - 2e^x\, dt dy$

Kinematical Quantities

For any timelike unit vector field V, one has

$$g(\cdot, \nabla V) = \frac{\Theta}{n} h + \sigma + \omega - g(\dot{V}, \cdot) \otimes g(V, \cdot)$$

with the following quantities

$\Theta = \operatorname{div} V$	(expansion)
$\dot{V} = \nabla_V V$	(acceleration)
$\sigma = \operatorname{sym} g(\cdot, \nabla V) + \frac{1}{2} g(\dot{V}, \cdot) \vee g(V, \cdot) - \frac{\Theta}{n} h$	(shear)
$\omega = \operatorname{asym} g(\cdot, \nabla V) + \frac{1}{2} g(\dot{V}, \cdot) \wedge g(V, \cdot)$	(vorticity)
$h = g + g(V, \cdot) \otimes g(V, \cdot)$	(projection tensor)

Applications of Kinematical Quantities

- Conjugate points, singularity theorems (via the Raychaudhuri equation).
- Existence of special timelike vector fields ($u = g(V, \cdot)$):

Symmetry		**Kinematical Quantities**
conformally stationary	$\Longleftrightarrow$	$\sigma = 0$, $d(\dot{u} - \frac{1}{n}\Theta u) = 0$
stationary	$\Longleftrightarrow$	$\sigma = 0$, $\Theta = 0$, $d\dot{u} = 0$
static	$\Longleftrightarrow$	$\sigma = 0$, $\Theta = 0$, $d\dot{u} = 0$, $\omega = 0$
Hubble isotropic	$\Longleftrightarrow$	$\sigma = 0$, $\dot{u} = 0$

- Relativistic fluid dynamics and cosmological models.
- **Causality theory?**

Kinematical Characterization of Conformally Stationary Spacetimes

Proposition. (Ehlers 1961, Hasse and Perlick 1987, et al.)
Let (M, g) be an $(n+1)$-dimensional spacetime and V an observer field on (M, g). Then there exists a conformal Killing vector field parallel to V iff V is shear-free and the *red-shift one-form*

$$\rho = g\left(\nabla_V V, \cdot\right) - \frac{\operatorname{div} V}{n} g\left(V, \cdot\right)$$

is closed.

A Metric Characterization of H-Isotropic Spacetimes

Proposition.

In an H-isotropic spacetime (M, g), there exist local coordinates $(t, x^1, \ldots, x^n) = (t, x^A)$ such that

$$
\begin{aligned}
g_{ij}(t, x^A) &= g_{0i}(0, x^A) g_{0j}(0, x^A) \\
&\quad + S^2(t, x^A) \left(g_{ij}(0, x^A) - g_{0i}(0, x^A) g_{0j}(0, x^A) \right) \\
&= V_i(x^A) V_j(x^A) \\
&\quad + S^2(t, x^A) \left(g_{ij}^{(0)}(x^A) - V_i(x^A) V_j(x^A) \right) \\
&= V_i(x^A) V_j(x^A) + S^2(t, x^A) h_{ij}(x^A) \, .
\end{aligned}
$$

A Metric Characterization of H-Isotropic Spacetimes

Proof. Choose a coordinate system that is comoving with respect to the observer field and integrate the resulting differential equations. This leads to

$$\begin{aligned} g_{ij}(t,x^A) &= g_{0i}(t,x^A)g_{0j}(t,x^A) + \\ & S^2(t,x^A)\Big(g_{ij}(0,x^A) - g_{0i}(0,x^A)g_{0j}(0,x^A)\Big) + \\ & 2\int_0^t \frac{\sigma_{ij}(\tau,x^A)}{S^2(\tau,x^A)}\,d\tau\ . \end{aligned}$$

Accounting for the kinematical constraints gives the result.

A Metric Characterization of Conformally Stationary Spacetimes

Proposition.

In a conformally stationary spacetime (M, g), there exist local coordinates $(t, x^1, \dots, x^n) = (t, x^A)$ such that

$$g = -dt^2 + V_A dx^A dt + \tilde{g}_{AB} dx^A dx^B$$

with

$$\tilde{g}_{AB} = S^2(t)\,(g_{AB}(0) + F_A(t)F_B(t) - F_A(t)\,V_B(0) - F_B(t)\,V_A(0)) \ ,$$

where

$$F_A(t) = \int_0^t \frac{\partial_A \log f(\tau)}{S(\tau)}\,d\tau \ .$$

Stably Causal Conformally Stationary Spacetimes

Proposition. (Dirmeier, P., Scherfner 2008)
If a spacetime (M, g) admits a timelike conformal vector field ξ for which the observer field $V = \frac{\xi}{\sqrt{-g(\xi,\xi)}}$ satisfies

$$g\left(\nabla_V V, \nabla_V V\right) < \frac{(\operatorname{div} V)^2}{n^2} ,$$

then (M, g) is stably causal.

Stably Causal Conformally Stationary Spacetimes

Proof. For the function $f = \sqrt{-g(\xi, \xi)}$ one readily computes

$$g(\nabla f, \cdot) = df = fd(\log f) = f\rho \ .$$

Thus, f satisfies the timelike eikonal inequality and defines a global time function iff

$$g(\nabla f, \nabla f) = f^2 \left(g\left(\nabla_V V, \nabla_V V\right) - \frac{(\mathrm{div}\ V)^2}{n^2} \right) < 0 \ .$$

Gödel-type spacetimes (Obukhov, Korotkii et al.)

Consider the manifold $M = \mathbb{R} \times M_0$ with metric

$$g = -dt^2 + 2a(t)\nu_A e^A dt + a(t)^2 \beta_{AB} e^A e^B \ ,$$

where ν_A, β_{AB} are constants with $\beta = (\beta_{AB})$ symmetric and invertible, e^A are invariant one-forms on the 3-dimensional homogeneous space sections $t = const$.

(M, g) admits the timelike conformal vector field $\xi = a\partial_t$ with conformal factor $\Phi = 2\frac{da}{dt}$, and we have e.g.

$$\begin{aligned} \operatorname{div} V &= 3\frac{1}{a}\frac{da}{dt} \ , \\ g\left(\nabla_V V, \cdot\right) &= \frac{da}{dt}\nu_A e^A \ . \end{aligned}$$

Causal Gödel-type Spacetimes

Proposition.

A Gödel-type spacetime with positive-definite β contains no smooth CTCs.

Proof. The global coordinate function t has to be periodic with respect to the curve parameter s of a CTC γ. Thus, there is some s_0 with $\frac{dt}{ds}(s_0) = 0$ and at this point

$$g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) = a^2 \beta_{AB} e^A \left(\frac{d\gamma}{dt}\right) e^B \left(\frac{d\gamma}{dt}\right) .$$

Example. For the Gödel spacetime, $\det \beta = -\frac{1}{2}$.

Local vs. Global

Problem.
Can the above argument be generalized? Under which conditions can a CTC be timelike homotopically transformed to be contained in one adapted coordinate patch?

Remark.
A Gödel-type spacetime with positive-definite β is actually stably causal. (t satisfies the timelike eikonal inequality.)

Globally-hyperbolic Gödel-type Spacetimes

Proposition. (Dirmeier 2008)

A Gödel-type spacetime with positive-definite β is even globally hyperbolic.

Proof. Consider the Finsler metric on $T(\{t_0\} \times M_0)$

$$\begin{aligned} F &= \sqrt{a^2(t_0)(\beta_{AB} + \nu_A \nu_B) e^A \otimes e^B} + a(t_0)\nu_A e^A \\ &= \sqrt{\tilde{g}} + b \qquad \text{(Fermat metric)}. \end{aligned}$$

Since $(\{t_0\} \times M_0, \tilde{g})$ is a homogeneous Riemannian space, it is complete. The splitting ensures that ξ is complete.

Globally-hyperbolic Gödel-type Spacetimes

Following [Javaloyes 2007], F is forward and backward complete if $\sup_{p \in M_0} \|b_p\| < 1$; which in turn implies that the $\{t_0\} \times M_0$ are Cauchy hypersurfaces. One has

$$\|b_p\| = (\beta + \nu \otimes \nu)^{AB} \nu_A \nu_B \ ,$$

and for diagonal β

$$\|b_p\| = \frac{\alpha}{\alpha + \det \beta} < 1$$

with

$$\alpha = \beta_{22}\beta_{33}\nu_1^2 + \beta_{11}\beta_{33}\nu_2^2 + \beta_{11}\beta_{22}\nu_3^2 \ .$$

Gödel Model with Expansion

Consider $M = \mathbb{R}^3$ with Lorentzian metric

$$g = -dt^2 - 2\exp(x)dtdy + S(t)^2 dx^2 - (2 - S(t)^2)\frac{1}{2}\exp(2x)dy^2$$

The unit vector field ∂_t is shear-free and has geodesic flow, and there are no CTCs entirely contained in the region $\{S(t) > \sqrt{2}\}$.

Since the Gödel spacetime ($S(t) \equiv 1$) is totally vicious and this is a stable property, there are CTCs entirely contained in some region $\{1 - \epsilon < S(t) < 1 + \epsilon\}$.

Non-totally Vicious?

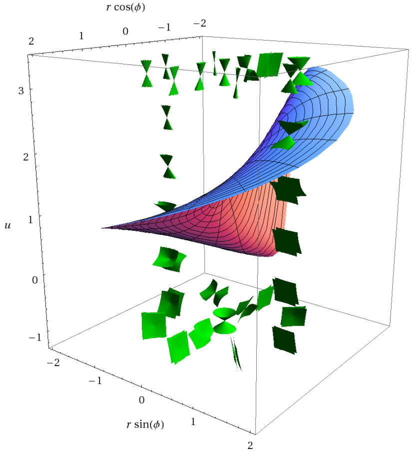


Image by T. Schönfeld

Thank you very much for your attention!