Conformal structures associated to generic rank 2 distributions on 5-manifolds.

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The distributions

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We consider generic rank 2 distributions in dimension 5, i.e. rank 2 subbundles $\mathcal{H} \subset TM$ such that sections of $\mathcal{H}$ and Lie brackets of two such sections span a rank 3 subbundle $[\mathcal{H}, \mathcal{H}]$ and Lie brackets of at most three sections span TM.

These distributions arise from ODE's of the form

$$z' = F(x, y, y', y'', z),$$

with $\frac{\partial^2 F}{\partial (y'')^2} \neq 0$ where y and z are functions of x.

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P. Nurowski:

A generic rank 2 distribution on a 5-manifold M determines a natural conformal structure of signature $(2, 3)$ on M.

Characterisation

A conformal class of signature $(2,3)$ metrics $[g]$ on M is canonically associated to a generic rank 2 distribution $\mathcal{D} \subset TM$ iff there exists a normal conformal Killing 2-form ϕ that is locally decomposable and satisfies the genericity condition

$$\phi \wedge \mu \wedge \rho \neq 0,$$

where

$$\begin{aligned}
\mu &:= \mathrm{tr}_{1,2} D\phi \in T^*M \\
\rho &:= -2\mathrm{tr}_{1,2} DD\phi + 4\mathrm{alt}(\mathrm{tr}_{1,3} DD\phi) + 3\mathrm{alt}(\mathrm{tr}_{2,3} DD\phi) \\
&\qquad + 24\mathrm{alt}(\mathrm{tr}_{1,3} P \otimes \phi) - 6\mathrm{tr}_{1,2} P \otimes \phi \in \Lambda^2 T^*M,
\end{aligned}$$

with D the Levi-Civita connection and P the Schouten tensor for a metric $g \in [g]$.

Decomposition of conformal Killing fields

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Choose $g \in [g_{\mathcal{H}}]$ and let ϕ be the normal conformal Killing 2-form characterizing the conformal structure. Then

$$\xi \mapsto \mathrm{tr}_{1,3}\mathrm{tr}_{2,4}(\phi \otimes D\xi + \frac{1}{6}\xi D\phi),$$

associates to a conformal Killing field $\xi \in \mathfrak{X}(M)$ its almost Einstein-scale part, and

$$\sigma \mapsto \frac{1}{3}\sigma \mathrm{tr}_{1,2} D\phi + \frac{2}{3}\mathrm{tr}_{2,3}\phi \otimes D\sigma$$

associates to an almost Einstein scale $\sigma \in \mathcal{E}[1]$ a conf. Killing field.

The Cartan geometries

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The results are derived employing methods from the theory of parabolic geometries; these are special kinds of Cartan geometries. A Cartan geometry of type (G, P) is given by

- a P-principal bundle $(\mathcal{G} \to M)$
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Conformal structures of signature $(2, 3)$ can be equivalently described as normal parabolic geometries of type $(SO(3,4), \tilde{P})$.

Split G_2

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The natural $GL(7, \mathbb{R})$-action on $\Lambda^3(\mathbb{R}^7)^*$ has 2 open orbits. The isotropy subgroup of a 3-form in either of these open orbits is a 14-dim. Lie group: for one orbit it is the compact real form, for the other open orbit it is the split real form G_2.

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Associated to $\Phi \in \Lambda^3(\mathbb{R}^7)^*$ is the bilinear map $\mathbb{R}^7 \times \mathbb{R}^7 \to \Lambda^7(\mathbb{R}^7)^*$,

$$(X, Y) \mapsto i_X\Phi \wedge i_Y\Phi \wedge \Phi,$$

which is non-degenerate iff Φ is contained in an open orbit. It determines a volume form and thus a bilinear form $\langle\ ,\ \rangle$, which has signature $(3, 4)$ iff the isotropy group of Φ is the split real form G_2.

Homogeneous models

Thus, we have a natural inclusion

$$G_2 \hookrightarrow SO(3,4).$$

Let $\tilde{P} \subset SO(3,4)$ be the stabilizer of an isotropic ray in $\mathbb{R}^7$ and $P = G_2 \cap \tilde{P}$. Then,

$$G_2/P \cong SO(3,4)/\tilde{P}.$$

This relates the homogeneous models of generic rank 2 distributions and conformal structures of signature $(2,3)$.

A Fefferman-type construction

Let $(\mathcal{G}, \omega)$ be a Cartan geometry of type (G_2, P). Then there exists a unique extension of ω to a Cartan connection $\tilde{\omega}$ on

$$\tilde{\mathcal{G}} = \mathcal{G} \times_P \tilde{P}.$$

This yields a functor mapping Cartan geometries of type (G_2, P) to Cartan geometries of type $(SO(3,4), \tilde{P})$. In particular, a generic 2-distribution $\mathcal{H}$ naturally determines a conformal class $[g]_{\mathcal{H}}$ of signature $(2, 3)$.

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Proposition 1

Normality of ω implies normality of $\tilde{\omega}$.

Conformal holonomy

Let $[g]$ be a conformal class of $(2,3)$-metrics and $(\tilde{\mathcal{G}}, \omega_N)$ the associated normal conformal Cartan geometry.

Extend ω_N to a $SO(3,4)$-principal bundle connection $\hat{\omega}_N$ on $\tilde{\mathcal{G}}' = \tilde{\mathcal{G}} \times_{\tilde{P}} SO(3,4)$. We introduce the conformal holonomy group as $\mathrm{Hol}([g]) = \mathrm{Hol}(\omega'_N)$.

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Corollary

The conformal holonomy of $[g]_{\mathcal{H}}$ associated to a 2-distribution $\mathcal{H}$ is contained in G_2.

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Corollary

The conformal holonomy of $[g]_{\mathcal{H}}$ associated to a 2-distribution $\mathcal{H}$ is contained in G_2.

Theorem 1

Conversely, if $\mathrm{Hol}([g]) \subset G_2$, then $[g]$ is naturally associated to a generic 2-distribution via a Fefferman-type construction.

Tractor 3-form

Let V be a $SO(3,4)$-representation. An associated bundle

$$\mathcal{V} = \tilde{\mathcal{G}}' \times_{SO(3,4)} V \cong \tilde{\mathcal{G}} \times_{\tilde{P}} V$$

is called a tractor bundle. $\tilde{\omega}'$ induces the tractor connection ∇.

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$$\mathcal{T} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{R}^7.$$

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Then, $\mathrm{Hol}([g]) = \mathrm{Hol}(\nabla) \subset G_2$ iff $\mathcal{T}$ carries a tractor 3-form $\Phi \in \Lambda^3\mathcal{T}^*$ such that $\nabla\Phi = 0$ and

$$i_X\Phi \wedge i_Y\Phi \wedge \Phi = \lambda \mathrm{h}(X,Y)\mathrm{vol}$$

for all $X, Y \in \Gamma(\mathcal{T})$ and nonzero $\lambda \in \mathbb{R}$.

First BGG-operators

Since $\tilde{P}$ preserves a filtration on V, any tractor bundle $\mathcal{V} = \tilde{\mathcal{G}} \times_{\tilde{P}} V$ is naturally filtered $\mathcal{V} \supset \mathcal{V}^1 \supset \cdots \supset \mathcal{V}^k$ and one has a projection

$$\Pi_0 : \mathcal{V} \to \mathcal{V}/\mathcal{V}^1.$$

The BGG construction yields a natural differential splitting

$$L_0 : \mathcal{V}/\mathcal{V}^1 \to \mathcal{V}.$$

Indeed, L_0 is the first map in a composition of maps defining the first BGG operator Θ_0, which gives an overdetermined system of PDE's

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Parallel sections:

If $s \in \Gamma(\mathcal{V})$, $\nabla s = 0$, then $s = L_0(\Pi_0(s))$ and $\Pi_0(s) \in \ker(\Theta_0)$.

Examples in conformal geometry

Standard tractors:

$\Pi_0 : \mathcal{T} \to \mathcal{E}[1]$ and, for $g \in [g]$, $\Theta_0(\sigma) = (D_a D_b \sigma + P_{ab}\sigma)_0$.
Elements in $\ker \Theta_0 \subset \mathcal{E}[1]$ are **almost Einstein scales aEs**$([g])$:

A function $f \in \mathrm{C}^\infty(M)$ is an almost Einstein scale for $g \in [g]$ if it is non-vanishing on an open dense subset U and $\hat{g} = f^{-2} g$ is Einstein on U, i.e., $\mathrm{Ric}(\hat{g}) = \lambda \hat{g}$.

Via Π_0 and L_0, there is a 1-1 correspondence between parallel sections $s \in \Gamma(\mathcal{T})$ and almost Einstein scales.

Examples in conformal geometry

Conformal adjoint tractors:

$\Pi_0 : \Lambda^2\mathcal{T} = \tilde{\mathcal{A}}M = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathfrak{so}(3,4) \to \mathfrak{X}(M).$

The 1st BGG gives the conformal Killing equation; elements in the kernel are thus **conformal Killing fields**, i.e.,

$$\mathbf{cKf}([g]) = \{\xi \in \mathfrak{X}(M) : \mathcal{L}_\xi g = e^{2f} g \text{ for } g \in [g],\ f \in \mathrm{C}^\infty(M)\}.$$

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Tractor 3-forms:

$\Pi_0 : \Lambda^3\mathcal{T} \to \Omega^2(M) \otimes \mathcal{E}[3].$

Elements in the kernel of the first BGG-operator are **conformal Killing** 2-**forms**. Parallel sections project to special solutions, named normal conformal Killing 2-forms by F. Leitner.

Characterisation

Let $[g]$ be a conformal structure of signature $(2,3)$. Then a parallel tractor 3-form Φ projects to a normal conformal Killing 2-form $\phi = \Pi_0(\Phi)$; it can be recovered as $\Phi = L_0(\phi)$.

Using this, we show that

$$i_X\Phi \wedge i_Y\Phi \wedge \Phi = \lambda \mathrm{h}(X, Y)\mathrm{vol}$$

iff ϕ is locally decomposable and

$$\phi \wedge \mu \wedge \rho \neq 0,$$

where

$$\begin{aligned}
\mu_a &:= g^{pq} D_p \phi_{qa} \\
\rho_{a_1 a_2} &:= -2D^p D_p \phi_{a_1 a_2} + 4D^p D_{[a_1} \phi_{|p|a_2]} + 3D_{[a_1} D^p \phi_{|p|a_2]} \\
&\quad + 24 P^p{}_{[a_1} \phi_{|p|a_2]} - 6P^p{}_p \phi_{a_1 a_2};
\end{aligned}$$

Decomposition of conformal Killing fields

Let $[g]_{\mathcal{H}}$ be a conformal structure associated to a 2-distribution $\mathcal{H}$. By naturality of the construction, every infinitesimal automorphism of $\mathcal{H}$, i.e., ξ in

$$\mathbf{sym}(\mathcal{H}) = \{ \ \xi \in \mathfrak{X}(M) : \mathcal{L}_\xi \eta \in \Gamma(\mathcal{H}) \ \forall \eta \in \Gamma(\mathcal{H}) \ \},$$

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is an infinitesimal automorphism for the associated conformal structure, i.e., a conformal Killing field.

We prove that conformal Killing fields of $[g]_{\mathcal{H}}$ indeed decompose into infinitesimal automorphisms of $\mathcal{H}$ and almost Einstein scales:

$$\mathbf{cKf}([g]_{\mathcal{H}}) = \mathbf{sym}(\mathcal{H}) \oplus \mathbf{aEs}([g]_{\mathcal{H}}).$$

Decomposition of conformal Killing fields

The curvature of a Cartan geometry $(\mathcal{G}, \omega)$ can be viewed as an element

$$K \in \Omega^2(M, \mathcal{A}M).$$

Via the natural projection

$$\Pi : \mathcal{A}M = \mathcal{G} \times_P \mathfrak{g} \to \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} = TM,$$

a section $s \in \Gamma(\mathcal{A}M)$ can be inserted into K.

Proposition (A. Cap)

Infinitesimal automorphisms of $(\mathcal{G}, \omega)$ can be identified with adjoint tractors $s \in \Gamma(\mathcal{A}M)$ such that

$$\hat{\nabla}^{\mathcal{A}} s = \nabla^{\mathcal{A}} s + K(\Pi(s), \cdot) = 0. \qquad (1)$$

Decomposition of conformal Killing fields

As a G_2-module

$$\mathfrak{so}(3,4) = \mathfrak{g}_2 \oplus \mathbb{R}^7.$$

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The decomposition is realized via the 3-form $\Phi \in \Lambda^3(\mathbb{R}^7)^*$ stabilized by G_2: The projection

$$i\Phi : \mathfrak{so}(3,4) = \Lambda^2\mathbb{R}^7 \to \mathbb{R}^7$$

is given by insertion of $\Lambda^2\mathbb{R}^7$ into Φ. Conversely, we include $\mathbb{R}^7$ via

$$i\Phi : \mathbb{R}^7 \to \Lambda^2\mathbb{R}^7 = \mathfrak{so}(3,4).$$

Decomposition of conformal Killing fields

Thus,

$$\tilde{\mathcal{A}}M = \mathcal{A}M \oplus \mathcal{T}.$$

Lemma

Consider $s = s_1 + s_2 \in \Gamma(\tilde{\mathcal{A}}M) = \Gamma(\mathcal{A}M) \oplus \Gamma(\mathcal{T})$. Then

$$\nabla^{\tilde{\mathcal{A}}} s + \tilde{K}(\Pi(s), \cdot) = 0$$

iff $\nabla^{\mathcal{A}} s_1 + K(\Pi(s_1), \cdot) = 0$ and $\nabla^{\mathcal{T}} s_2 = 0$.

Decomposition of conformal Killing fields

Theorem

Every conformal Killing field of $[g_{\mathcal{H}}]$ decomposes into a symmetry of the distribution $\mathcal{H}$ and an almost Einstein scale. Choose $g \in [g_{\mathcal{H}}]$ and let ϕ be the normal conformal Killing 2-form characterizing the conformal structure. Then,

$$\xi \mapsto \phi^{pq} D_p \xi_q + \frac{1}{6} \xi^p D^q \phi_{pq},$$

associates to a conformal Killing field $\xi \in \mathfrak{X}(M)$ its almost Einstein-scale part, and

$$\sigma \mapsto \frac{1}{3} \sigma D^p \phi_{pa} + \frac{2}{3} \phi_{ap} D^p \sigma$$

associates to an almost Einstein scale $\sigma \in \mathcal{E}[1]$ a conf. Killing field.