Spectral geometry of operators of Laplacian type

Georges Habib

1 Short Vita

Since 2008, Georges Habib has served as a professor of Mathematics at the Lebanese University in Beirut. In 2021, he earned his habilitation thesis from the University of Lorraine in France, subsequent to obtaining his Ph.D. from the same university in 2006. Following his doctoral studies, he undertook a postdoctoral position at the Max Planck Institute in Leipzig. Specializing in the realms of differential geometry and spectral theory, his primary research pursuits have led him to undertake multiple research visits across Europe. He is engaged in various projects funded by institutions including Humboldt, DAAD, Cèdre, Agence Universitaire de la Francophonie (AUF), the London Mathematical Society (LMS), and CNRS. Additionally, he has organized international conferences both in Lebanon and Europe.

2 Abstract

The primary goal of this project is to study the spectrum of certain differential operators, including the Laplacian and the Dirac operator, within the context of a Riemannian manifold. Among the questions that interest researchers in this field is the description of the spectrum of these operators in terms of geometric data in order to deduce topological obstructions. This relation between geometry, analysis, and topology has been the subject of intensive research over the past decades, resulting in several significant results. Within the framework of this project, various types of the Laplacian are considered with a special emphasis on understanding the behavior of the eigenvalues based on the geometric setting. Additionally, the project endeavors to address analytical problems seeking to understand their interpretations within the geometry and topology of the manifold.

3 Research Project

As previously highlighted, this project focuses on the special interplay between geometry, analysis, and topology. In particular, it considers different settings that are related to various analytical problems. During my stay at the Krupp Kolleg, I had the opportunity to complete and submit several ongoing projects and initiate new ones alongside researchers from different places. Accordingly, I will detail these projects, elucidating the specific context and scope of each one.

A Poincaré formula for differential forms and applications: This work is a collaboration with Nicolas Ginoux (University of Lorraine) and Simon Raulot (University of Rouen) and will be published in the journal SIGMA. In this paper, we study the general effect of the curvature operator of the interior of a manifold with a boundary on the topology and the geometry of its boundary. Inspired by questions arising in general relativity [1], Shi and Tam [11] considered a compact Riemannian spin manifold (M^n, g) with nonnegative scalar curvature such that its boundary ∂M is isometrically embedded into Euclidean space as a strictly convex hypersurface. They showed the following inequality:

$$\int_{\partial M} H d\mu_g \leq \int_{\partial M} H_0 d\mu_g \tag{1}$$

where H (resp. H_0) denotes the mean curvature of the boundary ∂M in M (resp. in the Euclidean space). Equality is attained if and only if the manifold M is isometric to a domain in the Euclidean space. In the same spirit, Hijazi and Montiel in [6] established a general integral inequality, which we refer to as *Poincaré type inequalities*, using an alternative method. More precisely, they proved that if (M^{n+1}, g) is a compact Riemannian manifold with a mean convex smooth boundary ∂M, then

$$\frac{n^2}{4}\int_{\partial M} H|\varphi|^2 d\mu_g \leq \int_{\partial M} \frac{|D\varphi|^2}{H} d\mu_g \tag{2}$$

for all spinor field φ defined on the boundary. Here, D is the Dirac operator acting on spinors on the boundary. As a consequence of this inequality, and when the boundary is isometrically immersed into another Riemannian manifold carrying a parallel spinor, they derive a new inequality similar to (1) which holds with fewer conditions and has interesting properties. Motivated by these results, Miao and Wang in [7] obtained an inequality as in (2) without assuming the spin condition, and where the Laplacian operator on functions is involved. The conditions required in this context are a lower bound on the Ricci curvature, and the technique uses the Reilly formula.

In this work, we generalize the results of Miao and Wang to differential forms by assuming general curvature conditions on the interior and on the boundary of the manifold. Here, we use the Hodge Laplace operator and the Reilly formula on differential forms established in [8]. The realization of equality in our estimate leads to various constraints on the differential form and the geometry of the manifold. In the particular case when the boundary is isometrically immersed in the Euclidean space, we deduce a rigidity result that extends the work of Miao and Wang. In a second step, we generalize the famous Ros inequality [9] to the setup of differential forms by assuming the existence of a parallel form on the manifold. This last inequality involves some curvature terms on the boundary.

A generalized Ricci-Hessian equation on Riemannian manifolds: This work is a collaboration with Nicolas Ginoux (University of Lorraine) and is currently submitted for publication. In this study, we are interested in classifying Riemannian manifolds (M^n, g) that support a smooth function satisfying the Ricci-Hessian equation, given by

$$\nabla^2 f = -f\mathrm{Ric}$$

where ∇^2 denotes the Hessian and Ric is the Ricci tensor of M. The motivation to study this equation comes from the skew-Killing spinor equation that was considered by the authors and Ines Kath in [4]. We show that, under some geometric conditions, the manifold M is isometric to the Riemannian product of a Ricci flat manifold with either the round sphere or the hyperbolic space. We point out that a full classification is not available yet which is the object of a future work.

A pseudodifferential analytic perspective on Getzler's rescaling: This work is a collaboration with Sylvie Paycha (University of Potsdam) and is currently submitted for publication. On a closed Riemannian manifold (M^n, g), the algebra of classical pseudodifferential operators defined on a vector bundle admits a unique trace, called the *Wodzicki residue* [13], built from a residue density as follows: Given a classical pseudodifferential operator Q acting on sections of a vector bundle E, the residue of Q is defined as the integral over M of the residue density $\mathrm{res}(\sigma(Q)(x,\cdot))dx^1 \wedge \ldots \wedge dx^n$ with

$$\mathrm{res}(\sigma(Q)(x,\cdot)) := \frac{1}{(2\pi)^n}\int_{|\xi|=1} \mathrm{tr}(\sigma_{-n}(Q)(x,\xi))d_S\xi.$$

Here, tr^E is the fibrewise trace on $\mathrm{End}(E)$ and $\sigma_{-n}(Q)(x,\xi))$ is the $(-n)$-th homogeneous part of the symbol at $(x,\xi) \in T^*M$. The Wodzicki residue extends beyond classical pseudifferential operators to the logarithm $\log_\theta Q$ of a pseudodifferential operator Q with Agmon angle θ. As the Wodzicki residue is local, the index of the Dirac operator D can be expressed in terms of this residue. Indeed, for a $\mathbb{Z}_2$-graded vector bundle $E = E^+ \oplus E^-$, the operator D is written as $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ so that $D^2 = D^-D^+ \oplus D^+D^-$. In this case,

$$\mathrm{Index}(D^+) = -\frac{1}{2}\mathrm{sres}(\log_\theta D^2) = -\frac{1}{2}\int_M \mathrm{sres}(\sigma(\log_\theta(D^2))(x,\cdot)).$$

Inspired by the approach adopted in [10], we revisit Geztler's rescaling in the context of index theory in the light of the logarithmic Wodzicki residue in this work. For a certain class of differential operators, including the Dirac operator, we express the logarithmic residue density in terms of another local density involving a limit of a family of differential operators built by *rescaling* the original operator at a fixed point. This new local density is defined on operators acting on differential forms. The rescaling, defined in terms of a local diffeomorphism and a contraction map, involves pulling back objects from the manifold to another manifold known as the *deformation to the normal cone.*

Eigenvalue estimates for the magnetic Hodge Laplacian on differential forms: This work is a collaboration with Michela Egidi (University of Rostock), Katie Gittins (University of Durham) and Norbert Peyerimhoff (University of Durham). It will be published in Journal of Spectral Theory. On a Riemannian manifold (M^n, g), the *classical magnetic Laplacian* associated with a smooth real 1-form α (called magnetic potential) acts on the space of smooth complex-valued functions and is given by

$$\Delta^\alpha := \delta^\alpha d^\alpha,$$

where $d^\alpha := d^M + i\alpha$ and δ^α is its L^2-adjoint. The magnetic Laplacian can be viewed as a first-order perturbation of the usual Laplace-Beltrami operator $\Delta^M := \delta^M d^M$. Here, δ^M is the L^2-adjoint of d^M. Several researchers have been interested in studying the spectral properties of the magnetic Laplacian, as it contains interesting information. We quote some of them. It is not difficult to check that the magnetic Laplacian has the property of gauge invariance, i.e., $\Delta^\alpha(e^{if}) = e^{if}\Delta^{\alpha+d^M f}$ for any smooth function f. In particular, the spectrum of Δ^α is equal to the spectrum of Δ^M when α is an exact form. Additionally, unlike the usual Laplacian, the first eigenvalue $\lambda_1^\alpha(M)$ is not necessarily zero, as shown in [12, Ex. 1]. This interesting property of the magnetic Laplacian was characterized by Shigekawa in [12, Thm. 4.2]. In the same spirit, the *diamagnetic inequality* compares the first eigenvalue of Δ^α to that of the Laplacian Δ^M, stating that

$$\lambda_1^\alpha(M) \geq \lambda_1(M)$$

with equality if and only if the magnetic potential α can be gauged away. The diamagnetic inequality holds true when the manifold M has a boundary, and the magnetic Laplacian is associated with certain boundary conditions such as Dirichlet or Robin. Hence, any lower bound for $\lambda_1(M)$ provides an estimate for $\lambda_1^\alpha(M)$. In [3, 5], the authors provide a magnetic Lichnerowicz type estimate for the first two eigenvalues using a Bochner-type formula, where the lower bound depends on the infinity norm of α. Additionally, in [2], the authors give an upper bound for the first eigenvalue depending on some distance.

In this work, we generalize the Hodge Laplacian $\delta^M d^M + d^M \delta^M$ on differential forms to the Hodge magnetic Laplacian by defining

$$\Delta^\alpha = \delta^\alpha d^\alpha + d^\alpha \delta^\alpha,$$

where $d^\alpha := d^M + i\alpha\wedge$ and $\delta^\alpha := \delta^M - i\alpha\lrcorner$. We study the spectral properties of this operator and prove the gauge invariance property. Additionally, we characterize the vanishing of the first eigenvalue, extending Shigekawa's result by assuming that the manifold carries a parallel form and the magnetic field α is a Killing one-form. It turns out that the diamagnetic inequality does not hold for the magnetic Hodge Laplacian, and we provide a counterexample on the round 3-sphere where α is the Reeb vector field that defines the Hopf fibration. Moreover, we extend the previous eigenvalue estimates to the setup of differential forms and derive a lower bound for the first eigenvalue on an embedded hypersurface of a Riemannian manifold.

4 Ongoing projects

During my stay, I initiated various projects, some of them are with new collaborators. In the following, I give a brief abstract of each one.

The Dirac operator on CR manifolds: This collaborative project with Felipe Leitner from the University of Greifswald focuses on the detailed analysis of the spectrum of the Kohn Dirac operator and the investigation of the twistor equation on CR manifolds. Our primary objective is to establish new estimates for the eigenvalues and to gain insights into the underlying geometry arising from the existence of these equations. Through rigorous mathematical analysis, we aim to deepen our understanding of the interplay between CR geometry and spectral properties, shedding light on the fundamental structures within this context.

Harmonic Unit vector fields: In collaboration with Andreas Savas-Halilaj and Konstantinos Vlachos from the University of Ioannina, we aim at studying the existence of harmonic vector fields on the round sphere. These vector fields are characterized as critical points of a specific *energy functional*. Our objective is to establish the necessary conditions that demonstrate the classification of these vector fields as Hopf vector fields.

Transversally formal metrics: This collaborative research, in conjunction with Ken Richardson from Texas Christian University and Robert Wolak from Jagiellonian University, is dedicated to the study of the concept of formal metrics within the realm of foliations. We refer to these metrics as *transversal formal metrics*. Through illustrative examples, we show that formality and transverse formality may not necessarily coincide within the context of Riemannian foliations.

Buckling and clamped plate operators on differential forms. This collaborative research, conducted in partnership with Fida Chami from the Lebanese University, Nicolas Ginoux from the University of Lorraine, Ola Makhoul from the Lebanese University, and Simon Raulot from the University of Rouen, introduces the buckling and clamped plate eigenvalue problems on differential forms, extending the corresponding concepts from functions. We establish various eigenvalue estimates that relate these differential forms and other essential eigenvalue problems, such as Dirichlet, Neumann, Steklov, and others.

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